More fun with Legendre...

$$a_{m+2} = \left[\frac{m(m+1) - n(n+1)}{(m+1)(m+2)}\right] a_m \quad (*)$$

* The recursion relation relates a_{m+2} to a_m
  $\therefore$ the even & odd polynomials de-couple:

$a_2 = \dots a_0$
$a_4 = \dots a_2 = \dots a_0$
$a_6 = \dots a_4 = \dots a_0$
$a_3 = \dots a_1$
$a_5 = \dots a_3 = \dots a_1$
$a_7 = \dots a_5 = \dots a_1$

ie. Solution = $a_0 \times$ {even polynomial}
$+ a_1 \times$ {odd polynomial}

* Sequence terminates for $m = n$
  $\Rightarrow a_{n+2} = 0$, then $a_{n+4} = 0$ etc

If n is even, we may therefore generate a polynomial sol. of degree n by setting $a_1 = 0$. Non-vanishing coeffs are $[a_0, a_2, a_4, \dots, a_n]$
Similarly if n is odd, we put $a_0 = 0$ & find an ~~odd~~ polyn. sol. with coeffs $[a_1, a_3, a_5, \dots, a_n]$

* The infinite set of solutions thereby generated have $\lambda = n(n+1)$ ordered as in Sturm-Liouville theory. These ARE the SL eigenfunctions

Expansion formula: $\quad f(x) = \sum_{n=0}^{\infty} d_n P_n(x), \quad d_n = \dfrac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\int_{-1}^{1} [P_n(x)]^2\, dx}$

$\sigma(x) = 1$

* n-even: $P_n = a_0(1 + \dots x^2 + \dots x^4 + \dots x^n)$
  n-odd: $P_n = a_1(x + \dots x^3 + \dots x^5 + \dots x^n)$

It is conventional to pick a_0 & a_1 so that $P_n(1) = 1$.

eg. $n=3$: $\quad P_3 = a_1 x + a_3 x^3, \quad a_3 = \dfrac{2-12}{6} a_1 \quad$ ($m=1, n=3$ in (*))
$= a_1\left(x - \frac{5}{3}x^3\right)$

$\therefore 1 = a_1\left(1 - \frac{5}{3}\right) \Rightarrow a_1 = -\frac{3}{2}$ & $\underline{P_3 = \frac{5}{2}x^3 - \frac{3}{2}x}$

eg. $n=4$

$P_4 = a_0 + a_2 x^2 + a_4 x^4$

$a_2 = -\dfrac{20}{2} a_0 \quad (m=0)$

$a_4 = \left(\dfrac{6-20}{12}\right) a_2 \quad (m=2)$

$a_2 = -10 a_0$

$a_4 = \dfrac{35}{3} a_0$

$= a_0\left(1 - 10x^2 + \frac{35}{3}x^4\right)$

$\Rightarrow 1 = a_0\left(1 - 10 + \frac{35}{3}\right) \longrightarrow \underline{P_4 = 3/8 - 15/4\, x^2 + 35/8\, x^4}$

$(a_0 = 3/8)$