More properties of associated Leg. funks.

* Simple examples

$m=n=1 \longrightarrow P_1^1 = -\sqrt{1-x^2}$
$\equiv -\sin\theta$

eg $u(1,\theta,\varphi) = \sin\theta\sin\varphi$ corresponds to $m=n=1$ term of general sol.
$\Rightarrow u(\rho,\theta,\varphi) = \rho\sin\theta\sin\varphi$

* $P_n^m = 0$ if $m>n$!

So $$u(\rho,\theta,\varphi) = \sum_{n=0}^{\infty} \rho^n \left[\frac{1}{2} a_{on} P_n + \sum_{m=1}^{n} (a_{mn}\cos m\varphi + b_{mn}\sin m\varphi) P_n^m \right]$$

* It can be shown that $$\int_{-1}^{1} P_{n_1}^m P_{n_2}^m \, dx = \begin{cases} 0 & n_1 \neq n_2 \\ \dfrac{2(n+m)!}{(2n+1)(n-m)!} \; (*) & n_1 = n_2 = n \end{cases}$$

(orthogonality condition of SL sols., with explicit integral evaluated)

* We can define $P_n^m(x)$ for $m<0$ by demanding
$$P_n^{-m} = \frac{(-1)^m (n-m)!}{(n+m)!} P_n^m$$
ie P_n^{-m} is just P_n^m upto a constant of proportionality ← useful to choose this so that a certain integral still holds, namely $(*)$

* In the general sol. a_{on}, a_{mn} & b_{mn} are arbitrary constants.

Also, $\cos m\varphi = \dfrac{e^{im\varphi} + e^{-im\varphi}}{2}$, $\sin m\varphi = \dfrac{e^{im\varphi} - e^{-im\varphi}}{2i}$

We can therefore manipulate the gen sol into
$$u(\rho,\theta,\varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn} Y_n^m(\theta,\varphi) \rho^n$$
← "Spherical harmonics"

with $$Y_n^m = P_n^m e^{im\varphi} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}$$ ← chosen so that we get this

This function satisfies

complex conjugate

$$\int_0^{2\pi}\int_{-1}^{1} Y_{n_1}^{m_1} (Y_{n_2}^{m_2})^* \, dx\, d\varphi = \begin{cases} 0 & \text{if } m_1 \neq m_2 \\ & \quad m_1 = m_2 \text{ but } n_1 \neq n_2 \\ 1 & \text{if } n_1 = n_2 \ \& \ m_1 = m_2 \end{cases}$$

follows from $(*)$ & φ-integral

Thus, $$A_{mn} = \int_0^{2\pi}\int_{-1}^{+1} (Y_n^m)^* f(\theta,\varphi) \, dx\, d\varphi$$