Spherical Harmonics

$$Y_n^m(\theta,\varphi) \equiv \sqrt{\frac{(2n+1)(n-m)!}{4\pi\,(n+m)!}}\; P_n^m(x=\cos\theta)\, e^{im\varphi}$$

Satisfy $\displaystyle\int_0^{2\pi}\int_{-1}^{1} |Y_n^m|^2\, dx\, d\varphi = 1$

since $\displaystyle\int_{-1}^{1} (P_n^m)^2 dx = \frac{2\,(n+m)!}{(2n+1)(n-m)!}$

$\displaystyle\int_0^{2\pi} e^{im\varphi} e^{-im\varphi} d\varphi = 2\pi$

Expansion: $\displaystyle f(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_n^m Y_n^m\,,\quad A_n^m = \int_0^{2\pi}\int_{-1}^{1} (Y_n^m)^* f\, dx\, d\varphi$

eg. $u_{tt} = \nabla^2 u$ for $u = u(\rho,\theta,\varphi,t)$

Plus
- * ICs in t
- * Regularity at $\theta = 0,\pi$ & $\rho = 0$
- * 2π-periodic in φ
- * additional condition in ρ such as $u(1,\theta,\varphi,t) = 0$

Pose
$$u = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} R_n^m(\rho,t)\, Y_n^m(\theta,\varphi)$$
(i.e. eigenfunk expansion)

The spherical harmonics satisfy
$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\,\frac{\partial Y_n^m}{\partial\theta}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_n^m}{\partial\varphi^2} = -n(n+1)\,Y_n^m$$

and so
$$\frac{\partial^2}{\partial t^2}R_n^m = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial R_n^m}{\partial\rho}\right) - \frac{n(n+1)}{\rho^2}R_n^m$$

* Quick & dirty: plug series into PDE

Cleaner & longer:
* multiply PDE by Y_n^m & integrate [projection]

If we also look for normal modes
with $R_n^m \propto e^{i\omega t}$ ... form of separable sol. anyway.

$$R_n^{m\,\prime\prime} + \frac{2}{\rho}R_n^{m\,\prime} - \frac{n(n+1)}{\rho^2}R_n^m + \omega^2 R_n^m = 0$$

Solution is $\displaystyle R_n^m = \text{const.}\; J_\nu(\rho\omega)\big/\rho^{1/2}$

with $\nu^2 = n(n+1) + \frac{1}{4}$ using gen. ODE Satisfied by Bessel funks

Last BC in ρ gives ω ...