Classification of 2nd-order, linear PDEs in two dimensions

General form of PDE:

principal part

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g$$

$a, b, \dots, g$ may dependent on x & y

* omit all terms except principal part
* set $\partial/\partial y = \xi_y$ & $\partial/\partial x = \xi_x$
* solve $a\lambda^2 + b\lambda + c = 0$ for $\lambda = \xi_x/\xi_y$

Parabolic PDE: $b^2 = 4ac$ (real, equal roots) eg. $u_t = u_{xx}$ heat eq.

Elliptic PDE: $b^2 < 4ac$ (complex roots) eg. $u_{xx} + u_{yy} = 0$ Laplace eq.

Hyperbolic PDE: $b^2 > 4ac$ (real roots) eg. $u_{tt} = u_{xx}$ wave eq.

All PDEs of 2nd order in 2D fall into one of these classes if linear.

Motivates a discussion of the three canonical examples,

$$u_t = u_{xx} \; ; \quad u_{xx} + u_{yy} = 0 \; ; \quad u_{tt} = u_{xx} .$$

Heat Eq $T = T(x,t)$ ← temperature of rod/wire at x & t

$\frac{d}{dt}\int_a^b \rho c_p T \, dx$ ← rate of change of heat contained in $[a,b]$

$\text{flux}_{(in)}\big|_{x=a} - \text{flux}_{(out)}\big|_{x=b}$

Must equal flux in/out & any net source or sink

Heat flux (in x-direction) is $-\kappa \frac{\partial T}{\partial x}$ by Fourier's law

Hence

density of sources & sinks

$$\frac{d}{dt}\int_a^b \rho c_p T \, dx = \underbrace{\int_a^b \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) dx}_{= \left[\kappa \frac{\partial T}{\partial x}\right]_a^b} + \int_a^b s(x,t) \, dx$$

→ heat eq. in integral form.