Using the definition of $\hat{f}(k)$ ...

$$u(x,t) = \int_{-\infty}^{\infty} f(\tilde{x})\,d\tilde{x} \times e_1(x-\tilde{x},t)$$

convolution integral! (interchanging the two integrals, using $\tilde{x}$ to distinguish integral for $\hat{f}$ from x).

$$e_1(x-\tilde{x},t) = \int_{-\infty}^{\infty} e^{-k^2t - ik(\tilde{x}-x)}\frac{dk}{2\pi}$$

Complete Square $\quad -t\left[k + \frac{i(\tilde{x}-x)}{2t}\right]^2 - \frac{(\tilde{x}-x)^2}{4t}$

Change Variable $\quad z = k + i(\tilde{x}-x)/2t$

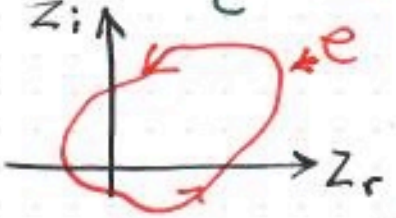
$$\rightarrow \quad e_1 = e^{-(\tilde{x}-x)^2/4t} \int_{-\infty + i(\tilde{x}-x)/2t}^{\infty + i(\tilde{x}-x)/2t} e^{-tz^2}\frac{dz}{2\pi}$$

$$\equiv \frac{e^{-(\tilde{x}-x)^2/4t}}{\sqrt{4\pi t}}$$

with the help of Cauchy's Theorem. (see below)

G is "Green's function" for this PDE.

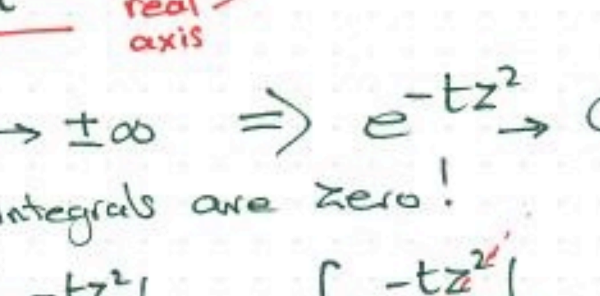
Cauchy: $\oint_{\mathcal{C}} f(z)\,dz = 0$ if $f(z)$ is analytic within the closed contour $\mathcal{C}$.



Convention: path is anti-clockwise

Here, we may use a rectangular box; the exponential e^{-tz^2} is analytic.

we need $\int_{\mathcal{C}_2} e^{-tz^2}dz$



loop needed to use Cauchy.

we know $\int_{\mathcal{C}_1} e^{-tz^2}dz = \sqrt{\pi/t}$ ← real axis

(z_i is finite)

Along $\mathcal{C}_3$ & $\mathcal{C}_4$, $z_r \to \pm\infty \Rightarrow e^{-tz^2} \to 0$

$\therefore$ these two integrals are zero!

Now, $$\int_{\mathcal{C}_1} e^{-tz^2}dz - \int_{\mathcal{C}_2} e^{-tz^2}dz + \int_{\mathcal{C}_3} e^{-tz^2}dz - \int_{\mathcal{C}_4} e^{-tz^2}dz = 0$$ by Cauchy

0   0

for the closed anti-clockwise loop, for $\mathcal{C}_2$ we proceed the wrong way $\Rightarrow$ switch sign

again going wrong way

Hence result