More transforms

$f(x)$	$\hat{f}(k)$
$f(x-a)$	$e^{-ika}\hat{f}(k)$
$e^{iax}f(x)$	$\hat{f}(k-a)$
$f(ax)$	$\frac{1}{\lvert a\rvert}\hat{f}(k/a)$
$H(x)e^{-ax}$	$\frac{1}{a+ik}$

} shifts

$$H(x)=\begin{cases}0 & x<0\\ 1 & x>0\end{cases}$$

"Heaviside" step funk.

Can attack inhomog. PDEs

eg. $u_t = u_{xx} + S(x) \quad -\infty < x < \infty$

FT of source

$u(x,0)=f(x) \xrightarrow{FT} \hat{u}(k,0)=\hat{f}(k)$ again

F.T. (ODE int) $\hat{u}_t = -k^2\hat{u} + \hat{S}(k)$ $\quad u\to 0$ for $x\to\pm\infty$.

$$\rightarrow \hat{u}(k,t) = \hat{f}(k)e^{-k^2t} + \hat{S}/k^2\left(1-e^{-k^2t}\right)$$ given $\hat{u}=\hat{f}$ at $t=0$.

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* Limitation on use of F.T. : spatially-dependent coeffs or nonlinear terms. eg. $(x\sin x)u$ or u^2 $\xrightarrow{F.T.}$ $\mathcal{F}\{x\sin x\, u\}$, $\mathcal{F}\{u^2\}$ (cannot relate to $\hat{u}$)

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* Can apply more FTs to deal with more space dimensions.
  eg. $u(x,y,t)$, $-\infty < x,y < \infty$
  But need the double transform $\hat{\hat{u}}(k,l,t) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)e^{-ikx-ily}\,dx\,dy$
  $(k\leftrightarrow x,\ l\leftrightarrow y)$ still need 3 variables!

$\boxed{\text{Laplace Transforms}}$

Def. $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt = \bar{f}(s)$

$\mathcal{L}\{df/dt\} = s\bar{f} - f(0)$ (action on deriv)

eg. $u_t = u - u_x, \quad 0\le t,x<\infty$
$u(x,0)=0, \quad u(0,t)=te^{-t}$

L.T. the PDE ↓ $\qquad$ LT the BC ↓

$\rightarrow s\bar{u} = \bar{u} - s\bar{u}_x \qquad \bar{u}(0,s) = \mathcal{L}\{te^{-t}\}$

ODE in x!

$\rightarrow \bar{u}(x,s) = \mathcal{L}\{te^{-t}\}e^{(1-s)x}$

Last, undo transform...

So $u(x,t) = \mathcal{L}^{-1}\left\{\mathcal{L}\{te^{-t}\}e^{(1-s)x}\right\}$

which amounts to computing some integrals.

$f(t)$	$\bar{f}(s)$
t^n	$n!/s^{n+1}$
e^{at}	$1/(s-a)$
$\cos\omega t$	$\omega/(s^2+\omega^2)$
$\sin\omega t$	$s/(s^2+\omega^2)$

Transform table

(from def. & integ. by parts)