Fourier Series

Continuous, periodic-funcs of period $2L$ can be represented as a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos A\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin A\,dx, \quad A \equiv \frac{n\pi x}{L}$$

If $f(x)$ has a jump discontinuity at $x = x_*$, the Fourier series may not converge to whatever value for $f(x_*)$ is chosen, but instead converges to

$$\frac{1}{2}\left[f^+ + f^-\right] \quad \text{where} \quad \begin{array}{l} f \to f^+ \text{ as } x \to x_* \text{ from } \underline{\text{right}} \\ f \to f^- \text{ as } x \to x_* \text{ from } \underline{\text{left}} \end{array}$$

(i.e. the average value of the limits)

Helpful integrals
$A = \frac{n\pi x}{L}$, $B = \frac{m\pi x}{L}$
n, m integers

$$\int_{-L}^{L} \sin A \sin B\,dx = \int_{-L}^{L} \cos A \cos B\,dx = \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \end{cases}$$

$$\int_{-L}^{L} \sin A \cos B\,dx = \int_{-L}^{L} \sin A\,dx = \int_{-L}^{L} \cos A\,dx = 0$$

{with $\int \cos\theta\,d\theta = \sin\theta$, $\int \sin\theta\,d\theta = -\cos\theta$, $\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$, $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

PDE solution: $\quad u = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

How do we know that u can be represented this way?

* $u = 0$ at $x = 0, \pi$
* $u_t = u_{xx}$ with $u(x,0) = f(x)$ determines $u(x,t)$ over $0 < x < \pi$
* $f(x)$ may not vanish at $x = 0, \pi$
  $\Rightarrow u(x,0)$ may have jump discontinuities.

Expect these to become smoothed out for arbitrarily small times by diffusion.

We use Fourier series theory to justify the PDE solution...