(Summary so far)

Separation of Variables

Put $u = X(x)T(t)$

* put in PDE & reorganize so that
$$\text{funk}(t) = \text{funk}(x)$$
$$\equiv \text{Separation constant}$$
* solve ODES
* assemble general sol.
* impose final conditions.

$$u_{tt} = u_{xx}, \quad u_x(0,t) = 0$$
$$u_x(\pi,t) = 0$$
$$u(x,0) = 0$$
$$u_t(x,0) = g(x)$$

* $\dfrac{X''}{X} = \dfrac{T''}{T} = -\lambda$

expect wavy sols to wave eq.

needed because $X' = 0$ at two positions.

Gives $X = A\cos nx$, $\lambda = n^2$

OR $X = \text{const}$, $\lambda = 0$.

Then, $T = C\sin nt$ OR $\text{const.} \times t$ $\Big\}$ since $T(0) = 0$

$$\therefore u = \frac{1}{2}a_0 t + \sum_{n=1}^{\infty} \frac{a_n}{n} \cos nx \sin nt$$

on choosing constants so that
$$u_t(x_0) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

i.e. the Fourier cosine series for the even, periodic extension of $g(x)$.

Expansion as eigen or basis functions.

* pose expansion at outset, paying attention to BCs.
* coeffs become time dependent; satisfying PDE corresponds to infinite set of ODEs.
* Apply final (Initial) Cond.

For this example, pose
$$u = \frac{1}{2}A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos nx$$
which satisfies BCs, & is justified by even, periodic extension. The ICs are

$$A_0(0) = A_n(0) = 0 \;\&$$
$$\left.\begin{aligned}\dot{A}_0(0) &= a_0 \\ \dot{A}_n(0) &= a_n\end{aligned}\right\}$$ Fourier cosine Series coeffs of even periodic extension of g.

Plug into PDE:
$$\frac{1}{2}\ddot{A}_0 + \sum_{n=1}^{\infty} \ddot{A}_n \cos nx = -\sum_{n=1}^{\infty} n^2 A_n \cos nx$$
$$\rightarrow \ddot{A}_n = -n^2 A_n, \quad \ddot{A}_0 = 0$$
$\rightarrow$ same solution as sep. of var.