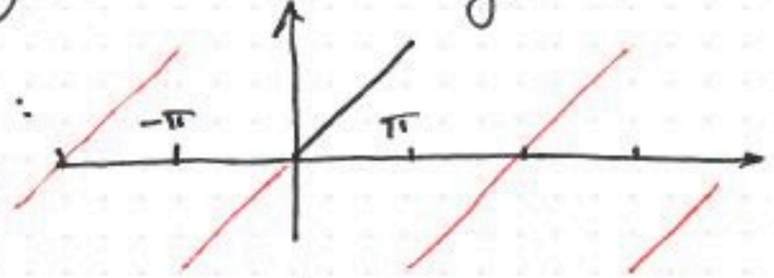
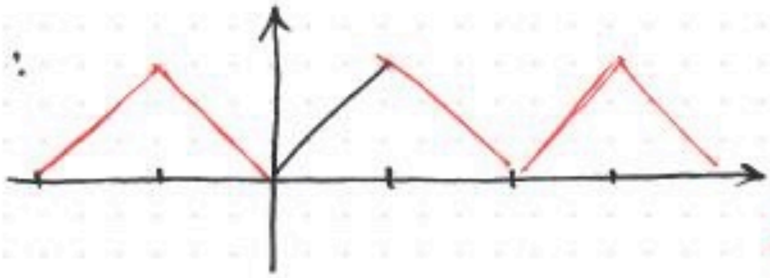
A cautionary note: e.g. $f(x)=x$ for $0<x<\pi$.

Odd, periodic extension: $(a_0=a_n=0)$



Has jumps for $x=\pm\pi, \pm 3\pi$ etc.

Even, periodic extension: $(b_n=0)$



Derivative has jumps for $x=0, \pm\pi, \pm 2\pi, \ldots$

Extensions generate functions with jumps in funk or derivative of it.

Can we differentiate the series ??

e.g. $f(x)=x$

$$= \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{odd, per. extension}$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx \equiv \frac{2}{n}(-1)^{n+1}$$

coeffs decreasing

coeffs not decreasing

$$\therefore \quad f(x) = 2\left(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \ldots\right)$$

$$f'(x) = 2\left(\cos x - \cos 2x + \cos 3x + \ldots\right)$$

but this should be ($f'=1$ except at jumps.)

e.g. $x=0 \longrightarrow \quad 2\,(1-1+1-1+1\ldots)$

$\rightarrow$ undefined.

So we cannot differentiate this series.

Instead, let's use projection (avoid differentiating series). Define

$$A_0(t) = \frac{2}{\pi}\int_0^{\pi} u(x,t)\,dx, \quad A_n(t) = \frac{2}{\pi}\int_0^{\pi} u(x,t)\cos nx\,dx$$

giving the series for u as employed earlier.

Take PDE, integrate in x (& multiply by $2/\pi$): $\frac{2}{\pi}\int_0^{\pi} u_{tt}\,dx = \frac{2}{\pi}\int_0^{x} u_{xx}\,dx$

$$\rightarrow \ddot{A}_0 = 0$$

Then PDE $\times \cos nx$:

$$\frac{2}{\pi}\int_0^{\pi} u_{tt}\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} u_{xx}\cos nx\,dx \rightarrow \ddot{A}_n = -n^2 A_n$$

after integrating by parts.