Laplace's Eq. for a disk with unit radius.

The problem: $\nabla^2 u = \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$ u 2π-periodic in θ

u regular as $r \to 0$

$u(1,\theta) = f(\theta)$.

F.S. for boundary funk. $f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\, d\theta, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \cos n\theta\, d\theta, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \sin n\theta\, d\theta$$

Separation of Variables $u = R(r)\Theta(\theta)$

$\rightarrow \frac{r}{R}(rR')' = -\frac{\Theta''}{\Theta} \longrightarrow$ sep. const. $\lambda = n^2$ (gives periodic funks for Θ)

ODES: $r^2 R'' + rR' - n^2 R = 0$

$\Theta'' + n^2\Theta = 0 \longrightarrow \Theta = \text{const.} \times \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}$ or const. if $n=0$.

Solution by a Fourier series expansion

Pose $u(r,\theta) = \frac{1}{2}A_0(r) + \sum_{n=1}^{+\infty}\left[A_n(r)\cos n\theta + B_n(r)\sin n\theta\right]$

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} u\, d\theta, \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u \cos n\theta\, d\theta, \quad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u \sin n\theta\, d\theta.$$

Plug in to PDE, match up terms (diff. of F.S. can be avoided by projection if required)

$\rightarrow (rA_0')' = 0, \quad r(rA_n')' - n^2 A_n = 0, \quad r(rB_n')' - n^2 B_n = 0$

i.e. same ODES as R-eq. (with $n=0$ for A_0)

Solution for R: $\underline{n=0}$ (A_0) $\Rightarrow (rR')' = 0 \rightarrow R = \text{const.} + \cancel{\text{const.} \ln r}$ not regular at $r=0$.

$\underline{n \neq 0}$ $(A_n \& B_n)$: Pose $R = Cr^{\alpha}$ (arbitrary const.) $\Rightarrow \alpha^2 = n^2$

$\therefore R = \text{const.} \times \begin{cases} r^n \\ \cancel{r^{-n}} \end{cases}$ irregular at $r=0$.

Hence,

$$u = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)$$

where constants have been chosen to ensure that $u(1,\theta) = f(\theta)$