Poisson's Solution (Summing the series)

Use $\hat{\theta}$ for integration variable to avoid confusion with θ of $u(r,\theta)$!

Given the defs. of a_0, a_n, b_n

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\hat{\theta})\,d\hat{\theta} + \sum_{n=1}^{\infty} \frac{r^n}{\pi}\int_{-\pi}^{\pi} f(\hat{\theta})\Big[\underbrace{\cos n\theta\cos n\hat{\theta} + \sin n\theta \sin n\hat{\theta}}_{\cos n(\theta-\hat{\theta})}\Big]\,d\hat{\theta}$$

But $\cos A = \dfrac{e^{iA}+e^{-iA}}{2}$, so the sum can be written as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\hat{\theta})\Big[\sum_{n=1}^{\infty}\Big\{\underbrace{r^n e^{in(\theta-\hat{\theta})}}_{[re^{i(\theta-\hat{\theta})}]^n = z^n} + \underbrace{r^n e^{-in(\theta-\hat{\theta})}}_{[re^{-i(\theta-\hat{\theta})}]^n = (z^*)^n}\Big\}\Big]\,d\hat{\theta}$$

$$\text{if } z = re^{i(\theta-\hat{\theta})}, \quad z^* = re^{-i(\theta-\hat{\theta})}$$

Now, $$\frac{1}{1-z} = 1 + \sum_{n=1}^{\infty} z^n, \quad \frac{1}{1-z^*} = 1 + \sum_{n=1}^{\infty} (z^*)^n$$

and so,

$$u = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\hat{\theta})\,d\hat{\theta} + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\hat{\theta})\left[\frac{1}{1-z} - 1 + \frac{1}{1-z^*} - 1\right]d\hat{\theta}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,f(\hat{\theta})\,d\hat{\theta}}{1+r^2-2r\cos(\theta-\hat{\theta})}$$

"Poisson's Solution"

* Started with an infinite series, coeffs each given by an integral (impractical except to build approximations after truncating)

* Summed the series to obtain a single integral

(Wow! as Jurgen Klopp would say)