Wave Equation : $u_{tt} = c^2 u_{xx}$

C = wavespeed (constant)

$u(0,t) = u(\pi,t) = 0$

$u(x,0) = f(x), \quad u_t(x,0) = g(x)$

Expansion as Fourier Sine series ...

$$\begin{Bmatrix} f \\ g \\ u \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} f_n \\ g_n \\ B_n(t) \end{Bmatrix} \sin nx, \quad \begin{Bmatrix} f_n \\ g_n \\ B_n \end{Bmatrix} = \frac{2}{\pi}\int_0^{\pi} \begin{Bmatrix} f \\ g \\ u \end{Bmatrix} \sin nx \, dx$$

Projection:

$$\frac{2}{\pi}\int_0^{\pi} (\text{PDE}) \sin nx \, dx \rightarrow \ddot{B}_n = -n^2 c^2 B_n$$

(after two integrations by parts on the RHS, & use of the BCs)

So $B_n = f_n \cos nct + \frac{g_n}{nc} \sin nct$

since $B_n(0) = f_n$ and $\dot{B}_n(0) = g_n$

So
$$u(x,t) = \sum_{n=1}^{\infty} \left(f_n \cos nct + \frac{g_n}{nc} \sin nct\right) \sin nx$$

$$= \sum_{n=1}^{\infty} \left[\frac{2}{\pi} \int_0^{\pi} f(\hat{x}) \, \sin nx \cos nct \, \sin n\hat{x} \, d\hat{x} + \frac{2}{\pi n c} \int_0^{\pi} g(\hat{x}) \, \sin nx \sin nct \, \sin n\hat{x} \, d\hat{x} \right]$$

$\frac{1}{2}\sin n(x+ct) + \frac{1}{2}\sin n(x-ct)$

$\frac{1}{2}\cos n(x-ct) - \frac{1}{2}\cos n(x+ct)$

Also,

$$f(z) = \sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^{\pi} f(\hat{x}) \sin n\hat{x} \, \sin nz \, d\hat{x}$$

integrate in z, replace f by g

$$\int g(z)\, dz = \sum_{n=1}^{\infty} \left[-\frac{2}{\pi n} \int_0^{\pi} g(\hat{x}) \sin n\hat{x} \, \cos nz \, d\hat{x} \right]$$

The two terms with f in u are therefore $\frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct)$

The two terms with g are integrals of $g(z)$ with either $z = x+ct$ or $z = x-ct$.

In summary, we may write the solution for $u(x,t)$ as

$$\frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\, dz$$

(equivalent to summing the series)

This is d'Alembert's solution to the wave eq.