



SEARCHING FOR NONLINEARITY IN INTERNAL TIDE WAVEBEAMS

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Background and mathematical model

Satellite altimetry has revealed that significant dissipation of the barotropic tide occurs in the deep ocean in the vicinity of large topographic features (*e.g.* Egbert & Ray, 2000, *Nature*, **405**, 775). This has been attributed to the generation of internal gravity waves as the tide flows over such obstacles.

Linear inviscid theory of the tidally generated gravity waves indicates that these disturbance can take the form of intense wave beams propagating away from the points on the topography where the slope of the radiating gravity waves matches the slope of the bottom surface (*e.g.* Balmforth, Ierley & Young, 2002, *JPO*, **32**, 2900).

The perturbations to the density gradient induced by these beams are formally singular in inviscid theory, suggesting that they may immediately behave nonlinearly, suffering shear or other kinds of instability, or creating density inversions that overturn the stratification.

Linear, two-dimensional, inviscid gravity waves

The problem is formulated in terms of a streamfunction and buoyancy (density) perturbation,

$$(u, w) = \text{Re}(-\psi_z, \psi_x)e^{-i\omega t} \quad \text{and} \quad b = \text{Re} \frac{N^2}{i\omega} \psi_x e^{-i\omega t}$$

where ω is the tidal frequency and N the buoyancy frequency.

The internal wave equation is

$$\left(\frac{N^2}{\omega^2} - 1\right) \frac{\partial^2 \psi}{\partial z^2} = \frac{\partial^2 \psi}{\partial x^2}$$

Thus waves travel at an angle $\tan^{-1} \sqrt{N^2/\omega^2 - 1}$ to the horizontal; the phase and group velocities are orthogonal.

Boundary conditions and solution technique

On the topography:

$$\psi(x, h(x)) = Uh(x)$$

For an infinitely deep ocean, the waves generated at an obstacle propagate off to $z \rightarrow \infty$, and we impose an outgoing radiation condition.

We distribute sources with density, $\Gamma(x)$, along the topography and use a Green function, $G(x, x', z, z')$, to write:

$$\psi(x, z) = \int G(x, x', z, h(x')) \Gamma(x') dx'$$

(*cf.* Petrelis, Llewellyn Smith & Young, 2007, *JPO*, **36**, 1053). The lower boundary condition then furnishes an integral equation for $\Gamma(x)$.

We use periodic topography in x : either a sinusoid or a “periodic Gaussian”,

$$h(x) = h_0 \times \begin{cases} \sin(kx) \\ \exp[-\gamma(1 - \cos kx)] \end{cases}$$

where γ is a separation parameter ($\gamma \rightarrow 0$ recovers the sinusoidal profile, and $\gamma \rightarrow \infty$ gives an isolated Gaussian).

Far-field representation

Above the crests of the topography, the Green function can be simplified and the solution written in the form,

$$\psi(x, z) = \sum_{p=1}^{\infty} \frac{\Gamma_p}{p} e^{-ip\mu k z} + \text{constant},$$

where $\mu = \sqrt{N^2/\omega^2 - 1}$ and the γ_p 's are certain integrals of $\Gamma(x)$.

Assuming the viscosity to be small, we can improve this solution to account for viscous smoothing of the wave beams using perturbation theory. This replaces the factor, $e^{-ip\mu k z}$, with

$$\exp \left[-ip\mu k - \frac{\nu k^2 N^4}{\omega^3 (N^2 - \omega^2)} \right] z$$

The wave structures are then no longer singular and can be compared with experiments.

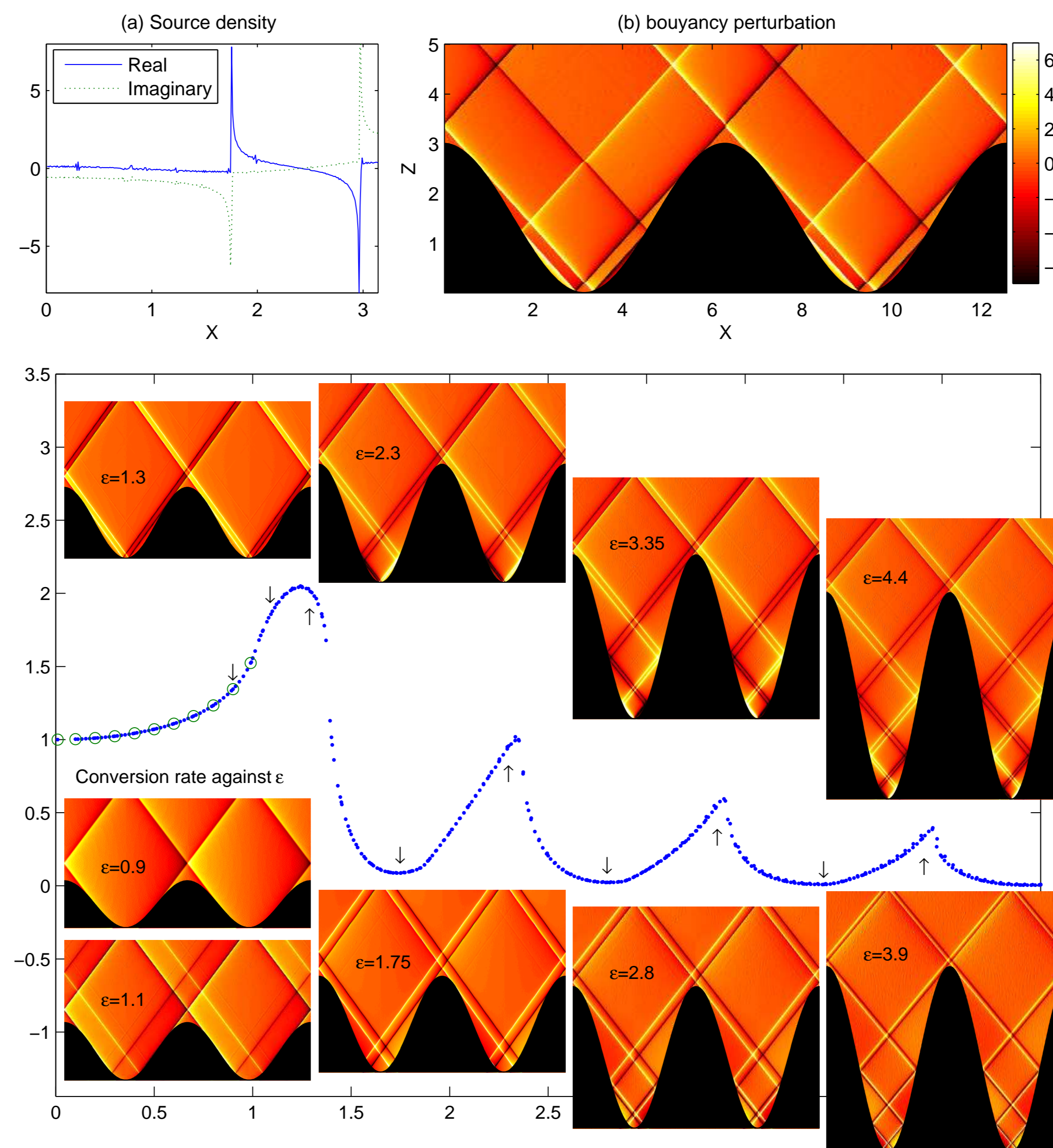


FIGURE 1: Beam generation and interference for sinusoidal topography, and the (dimensionless) energy conversion rate *vs.* ϵ , the ratio of topographic slope to wave slope.

The wave beams

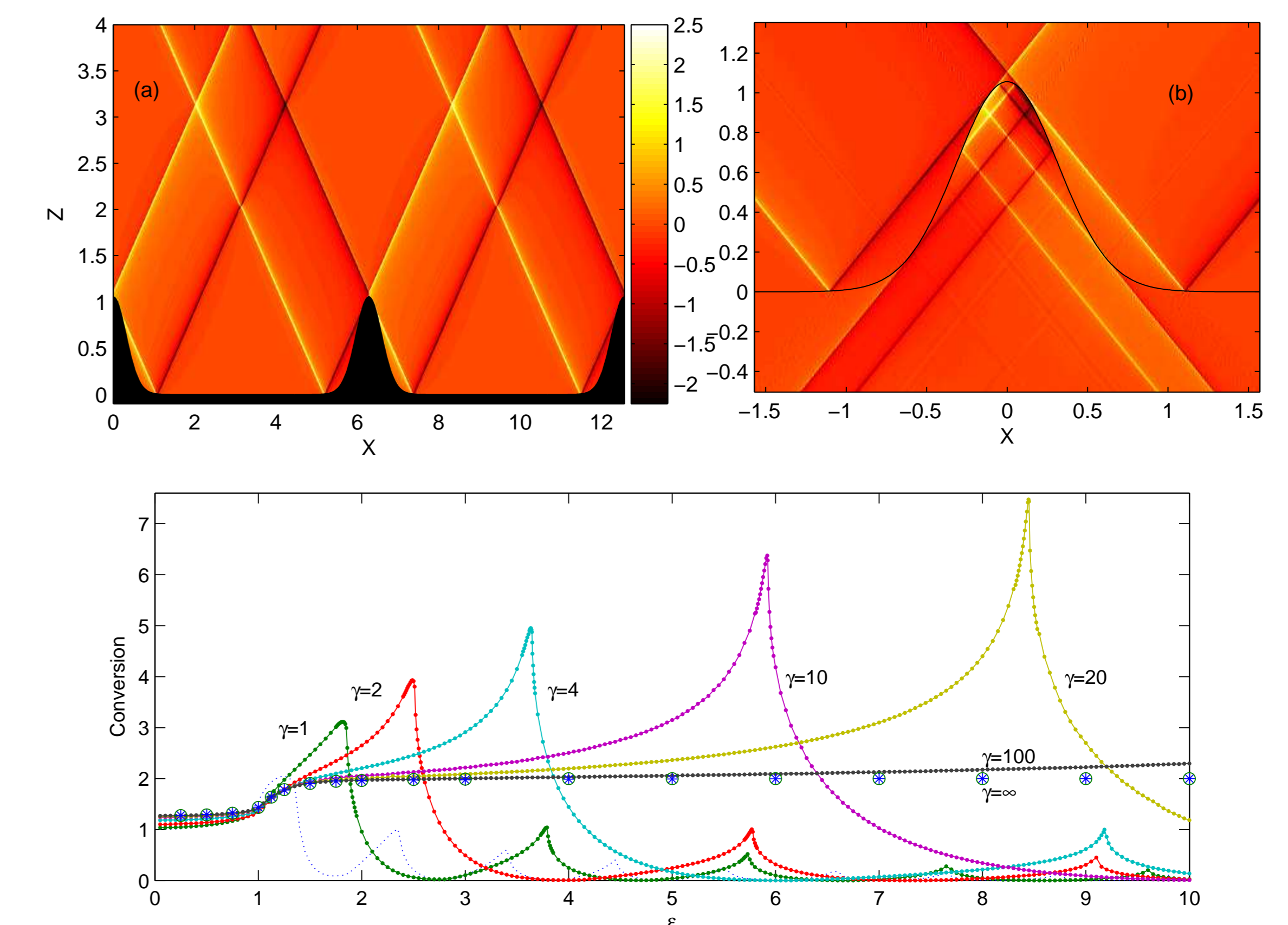


FIGURE 2: Beam generation above periodic Gaussian ridges (and trenches), and the conversion rate *vs.* ϵ for different bump separations, γ .

Wavebeams leave the topography at the critical slopes; the beams scatter either up or down and can interfere with one another constructively or destructively. This leads to an energy conversion rate with a complicated dependence on the topographic height (or relative maximum slope, ϵ ; *cf.* Nycander, 2006, *JFM*, **567**, 415).

The experiment

The experiment consists of a tank of stratified salty fluid, with an obstacle on the bottom that is dragged back and forth sinusoidally with a given frequency and amplitude. The digital Schlieren technique (Sutherland, Dalziel, Hughes & Linden, 1999, *JFM*, **390**, 93) is used to image the internal wave field.

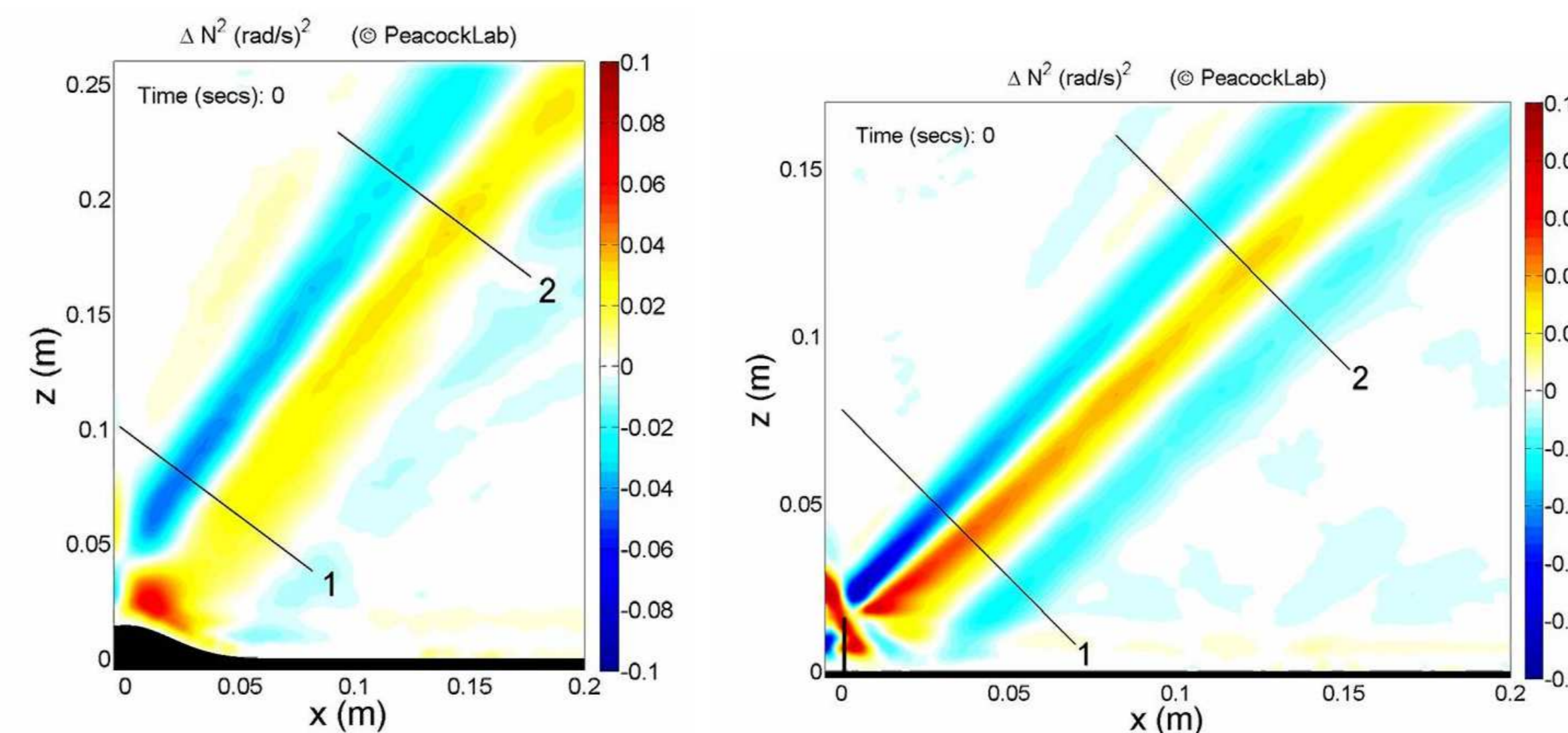


FIGURE 3: A gaussian bump and the limiting knife edge.

The experiment runs at relatively low Reynolds number (of order 100). As a result, viscosity is crucial in smoothing the wave beams and their amplitude becomes relatively weak (compare the viscous and inviscid curves). Consequently, we have yet to observe any secondary instabilities or overturning in the wavebeams.

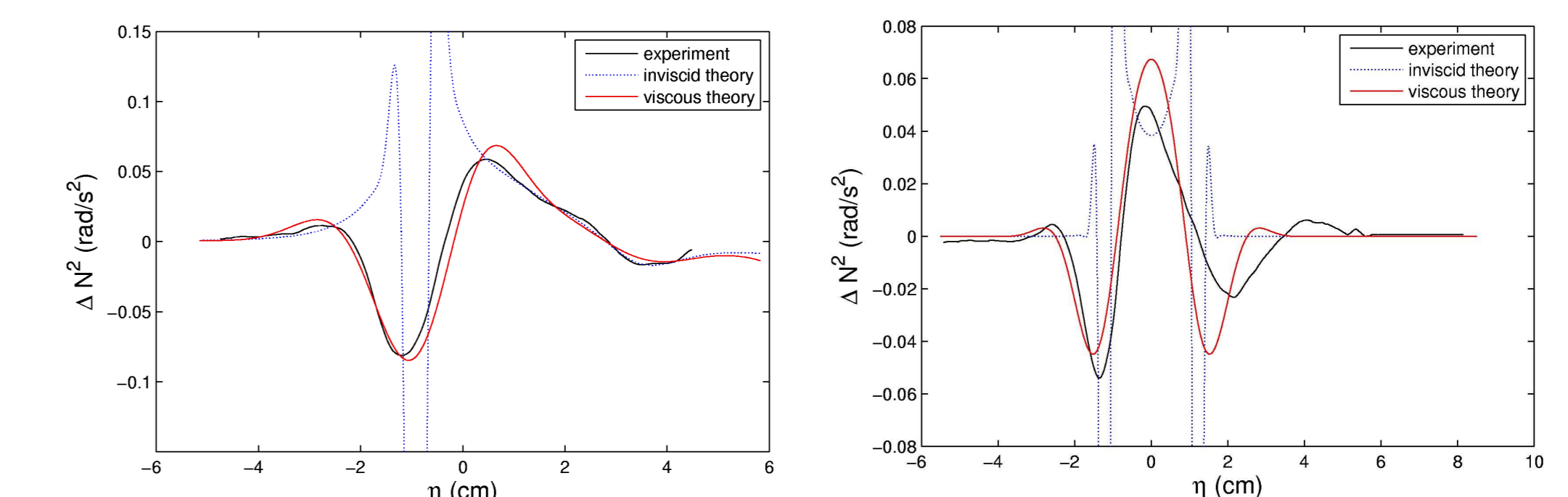


FIGURE 4: Comparison of theory and experiment.

We have, however, observed the generation of weak beams at twice the tidal frequency (the second harmonic).

We are still in search of more significant nonlinear effects...

