The Bingham-Rayleigh-Bénard Problem

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Jeffreys' Porridge Problem

"If porridge is cooked in a single saucepan and not stirred it will burn at the bottom. It can still be poured – it is still liquid, but at a certain stage of stickiness convection currents can be prevented even when the bottom is some hundreds of degrees hotter than the top." (Jeffreys 1957, Sir H. Jeffreys Papers C17)

Thermal convection in viscoplastic fluid is important in many industrial and geophysical applications. Jeffreys (1952, ***The Earth***, C.U.P.) and other geophysicists in thinking of convection in the Earth's mantle, were among the first to appreciate that a finite fluid strength would substantially affect convection (*e.g.* Griggs 1939, ***Am. J. Sci.*** 237, 611; Orowan 1965, ***Phil. Trans. Roy. Soc. A***, 258, 284). Orowan showed particular insight, suggesting that thermal convection would not readily initiate if the fluid has a yield stress, but that the Newtonian solution is a reasonable approximation once convection is underway.

Thermal convection in Newtonian fluids (the Rayleigh-Bénard problem) is a classic example of instability theory and pattern formation. Here we map out the corresponding problem for viscoplastic fluids. Unlike in the Newtonian fluid, the motionless conduction state is linearly stable for all Rayleigh number because the yield stress cannot be overcome by an infinitesimal perturbation (Zhang, Frigaard & Vola 2006, ***J.F.M.***, 566, 389). Can fluid microstructure be broken with a finite kick to initiate convection?

Mathematical model

In the dimensionless formulation of the convection problem, the fluid depth, d, and thermal conductivity, κ, are used to build units for length, speed and time, and the temperature difference across the plates scales temperatures.
In terms of a streamfunction, $\psi(x,z,t)$, satisfying $(u,w)=(-\psi_z,\psi_x)$, and a temperature perturbation, $\theta(x,z,t)$, the equations are, assuming large Prandtl number, $\nu/\kappa \gg 1$,

$$0=\nabla^4\psi + R\theta_x + \mathcal{B}$$

$$\theta_t + \psi_x\theta_z - \psi_z\theta_x = \nabla^2\theta + \psi_x,$$

R = Rayleigh number, ν = kinematic viscosity
$\mathcal{B}$ denotes the contribution from the non-Newtonian part of the stresses, $\hat\tau_{jk}$:

$$\mathcal{B}=\frac{\partial^2\hat\tau_{xz}}{\partial x^2}-\frac{\partial^2\hat\tau_{xz}}{\partial z^2}-2\frac{\partial^2\hat\tau_{xx}}{\partial x\partial z},$$

The Bingham Fluid

Dimensional deviatoric stresses : $\boldsymbol{\tau}=\left(\rho\nu+\frac{\tau_Y}{\dot\gamma}\right)\dot{\boldsymbol{\gamma}}, \quad \text{if } \tau_Y<\sqrt{\sum_{j,k}\tau_{jk}^2/2},$

and $\dot\gamma_{jk}=0$ otherwise, where the deformation rates are

$$\dot{\boldsymbol{\gamma}}=\begin{pmatrix}-2\psi_{xz} & \psi_{xx}-\psi_{zz}\\ \psi_{xx}-\psi_{zz} & 2\psi_{xz}\end{pmatrix}, \qquad \dot\gamma=\sqrt{4\psi_{xz}^2+(\psi_{xx}-\psi_{zz})^2}.$$

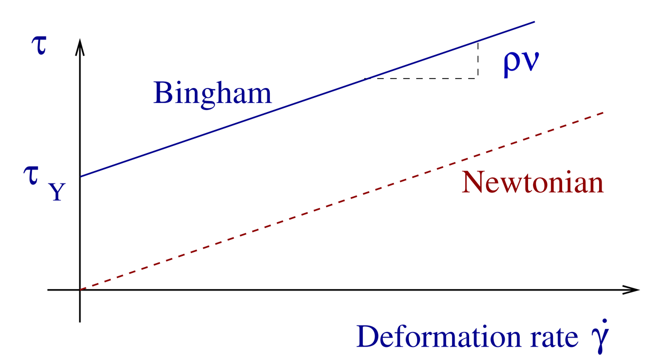

Fig. 1: The Bingham model

ν = kinematic viscosity,
τ_y = yield stress. ρ = density.

Dimensionless yield stress : $B=\frac{\tau_Y d^2}{\rho\nu\kappa}.$

The dimensionless yield stresses, $\hat\tau_{jk}=B\dot\gamma_{jk}/\dot\gamma$, lead to the non-Newtonian contribution,

$$\mathcal{B}=B\left[4\left(\frac{\psi_{xz}}{\dot\gamma}\right)_{xz}+(\partial_x^2-\partial_z^2)\left(\frac{\psi_{xx}-\psi_{zz}}{\dot\gamma}\right)\right].$$

Weakly nonlinear and non-Newtonian convection

Asymptotic expansion near onset (*cf.* Malkus & Veronis 1958, ***J.F.M.***, 4, 255):

$$\frac{\partial}{\partial t}=\epsilon^2\frac{\partial}{\partial T}, \qquad R=R_c+\epsilon^2 R_2, \qquad B=\epsilon^3 B_3, \qquad \mathcal{B}=\epsilon^3\mathcal{B}_3,$$

$$\psi=\epsilon\psi_1+\epsilon^2\psi_2+\epsilon^3\psi_3+..., \qquad \theta=\epsilon\theta_1+\epsilon^2\theta_2+\epsilon^3\theta_3+...$$

R_c is the critical Rayleigh number. The Non-Newtonian stresses are scaled by ϵ^3 to bring them into the amplitude equation for weakly nonlinear convection.
Leading-order solution (the most unstable, linear normal mode):

$$\psi_1=A(T)\sin kx\sin\pi z, \qquad \theta_1=\frac{1}{6}\pi A(T)\cos kx\sin\pi z,$$

k = critical wavenumber $(\pi/\sqrt{2})$.
Continuing on, eventually an amplitude equation falls out:

$$\frac{9\pi^2}{2}A_T=R_2A-\frac{9\pi^4}{32}A^3-\Gamma\,\mathrm{sgn}(A)$$

with $\Gamma\approx 78B_3$.

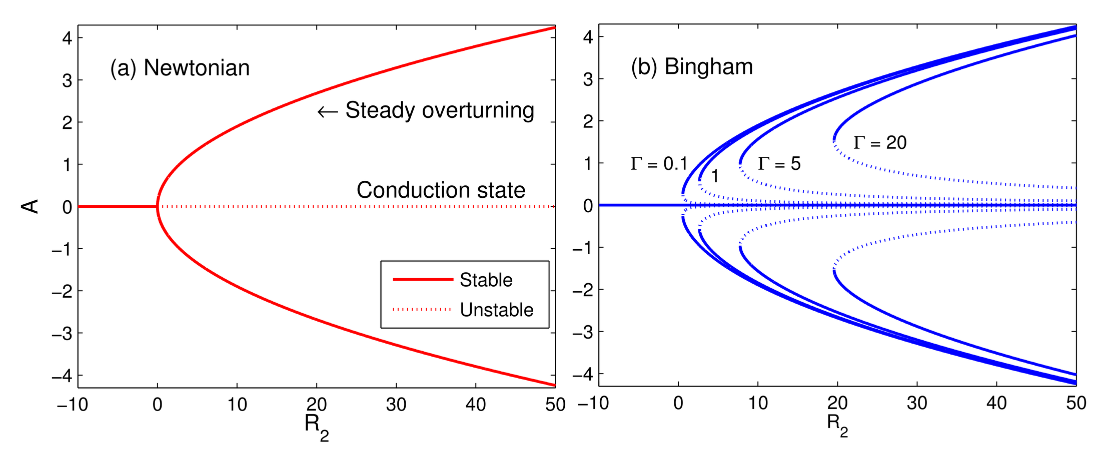

Fig. 2: Steady solutions to the amplitude equation.

Analytical and unstable, short-wavelength solutions

Along the lower branch of unstable solutions, there are analytical solutions with $k \gg 1$.

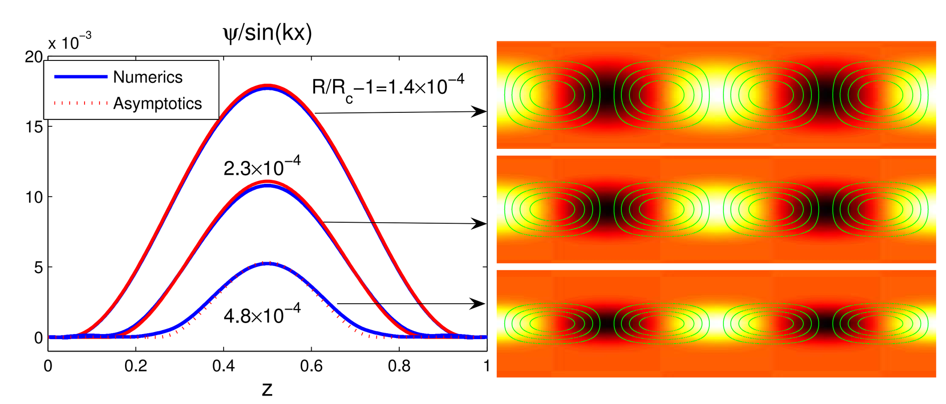

Fig. 3: Short-wavelength solutions in the form of localized layers: $k=10^3$, $B=1$.

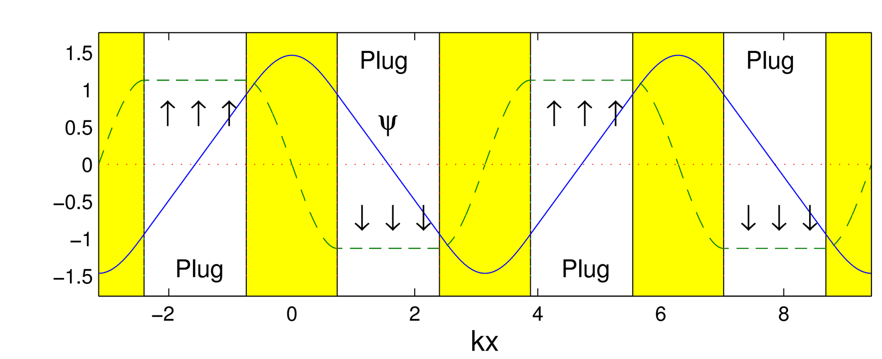

Fig. 4: Viscoplastic elevators (Gershuni & Zhukhovitski 1973, Sov. Phys. Dokl. 18, 36)

Although these solutions are unstable, they give estimates of the strength of the kick needed to initiate convection in the fully nonlinear regime.

Fully nonlinear steady solutions

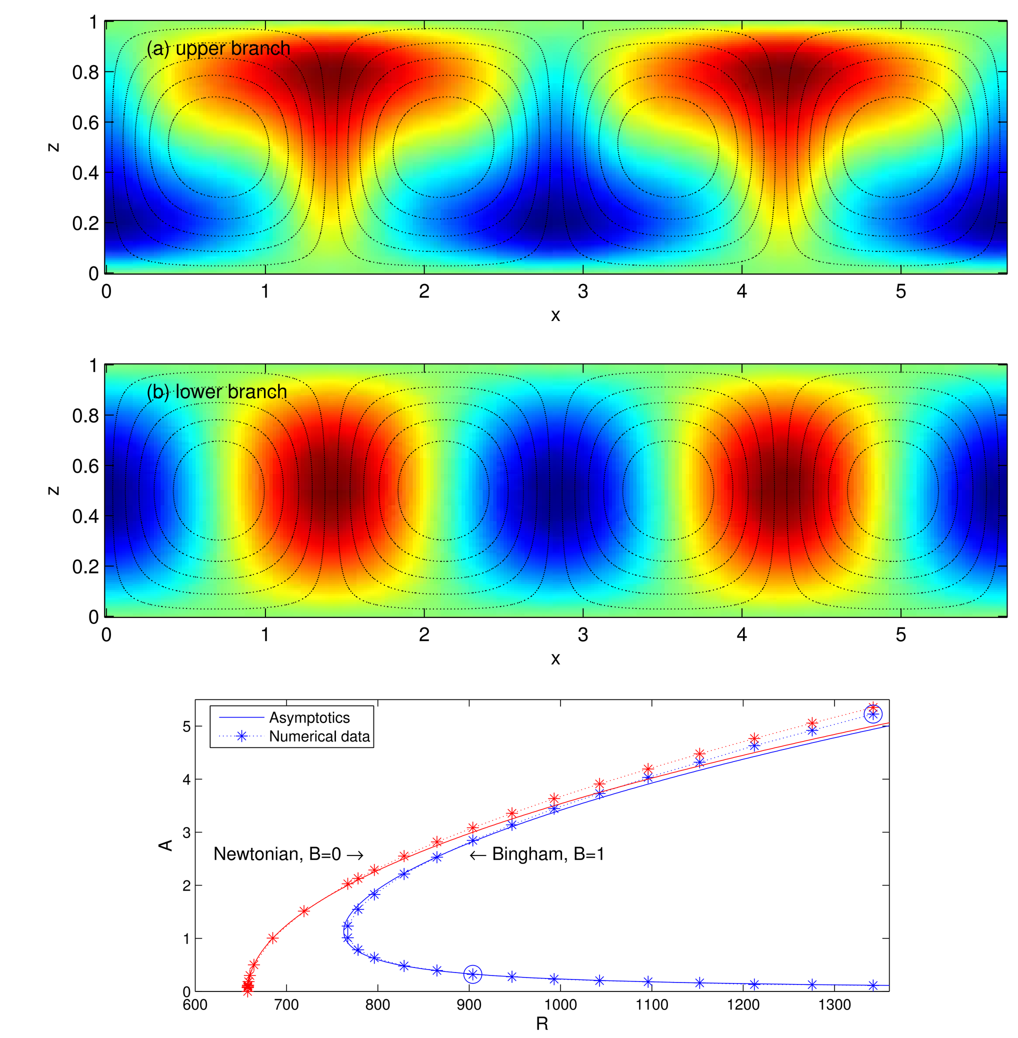

Fig. 5: Numerical solutions for $B=1$ along the upper and lower branches at the circles marked in the third plot. The colors indicate θ, and the contours show ψ.

Interpretation:
- yield strength suppresses convective instability
- a sufficiently strong perturbation can kick the fluid layer into convection
- once initiated, viscoplastic convection is much like the Newtonian relative

Experiments and geological applications

- The greater the yield strength, the larger the perturbation required to initiate convection.
- The conditions (temperature difference and lengthscale) required to initiate convection can be quite different from those required to perpetuate convection.

These conclusions were confirmed in an experiment in which Carbopol (a prototypical viscoplastic fluid) was heated from below: at low concentration (yield stress) convection could be initiated by ambient or forced perturbations; at high concentration, layer would never convect.

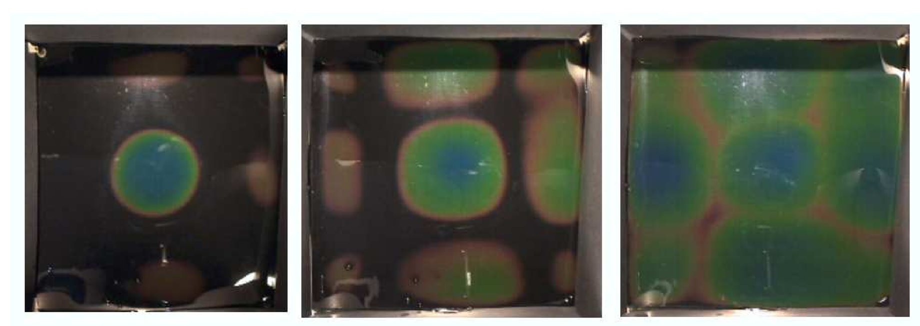

Fig. 6: Developing plumes revealed by a liquid crystal sheet above heated Carbopol.

- Applications include: ice slurries in refrigeration systems and around extraterrestrial bodies, drilling muds and gels, various magma structures and mud volcanoes.
- Understanding the role of crystals in magma convection has implications for the generation of economic ore deposits as well as volcanic eruption triggers and magma degassing
- Crystal-rich magmas are generally considered to be viscoplastic. As a magma cools, the crystal content, and thus the yield strength increases.
- To avoid the constraint imposed by a yield stress, convection may begin earlier when there were fewer crystals, or could be limited to the hotter parts of a magma chamber.