Asymptotics final

Answer as much as you can. May the force be with you.

1. For $z \gg 1$, find the leading-order approximation to the integral,

$$I = \int_0^1 \exp[iz\sin^2(\pi x^q)]\, dx,$$

where $q > 0$.

2(a). Find the coefficients of the $\xi \gg 1$ asymptotic approximation,

$$I = \int_0^\xi \exp\left(-\frac{\alpha}{t}\right)\frac{dt}{t} \sim a_0\ \log\xi + a_1 + a_2\xi^{-1}.$$

Show that $G = CI(\xi) + 2I(\xi/2) - e^{-\alpha\xi^{-1}}I(\xi)$, where C is arbitrary, solves

$$\xi^3 G'' + \xi(\xi - \alpha)G' = \alpha I e^{-\alpha\xi^{-1}}, \qquad G \to 0 \text{ as } \xi \to 0$$

and hence determine the coefficients of $G \sim b_0 \log\xi + b_1 + b_2\xi^{-1}\log\xi$ for $\xi \gg 1$.
2(b). The function $f(x)$ satisfies the equation,

$$x^2 f_{xx} + x f_x = \epsilon\pi f f_x$$

in $x \le 1$, with $\epsilon > 0$ and the boundary conditions, $f = 0$ on $x = 1$ and $f \to 1$ as $x \to 0$. Obtain an asymptotic expansion for f at fixed x, in the asymptotic sequence $(\delta, \delta^2, ...)$, where $\epsilon \ll \delta^2 \ll 1$. Then find an expansion for f at fixed $\xi = \epsilon^{-1}x$, in the sequence, $(1, \delta, \delta^2, ...)$. Match the two solutions for f, determining $\delta(\epsilon)$ along the way.

3. Using multiple scales, find the leading-order asymptotic approximation, valid for $t = O(\epsilon^{-1})$ to the solution of the equations,

$$\ddot{x} + x - y = \epsilon(x - |\dot{x}|), \qquad \dot{y} = \epsilon\left[(x - y)\sin t - y\right], \qquad x(0) = 1, \qquad \dot{x}(0) = 0, \qquad y(0) = 0.$$

4. Using the WKB method, provide an approximation for the eigenvalue, λ, of the problem

$$y'' + \lambda y f(x) = 0, \qquad y(0) = y(\pi) = 0,,$$

where $f(x) \to cx^q$ for $x \to 0$, with c and q positive constants, $f(\pi/2) = 0$, $f'(\pi/2) \ne 0$, $f > 0$ for $0 < x < \pi/2$, and $f < 0$ for $\pi/2 < x < \pi$. In the case $f(x) = x\cos x$, compare your results with the lowest eigenvalues computed numerically: $|\lambda| \approx 1.511$, 5.216, 8.423, 20.954 and 33.145.

Note that the WKB approximation to $y'' + f(x)y = 0$ is

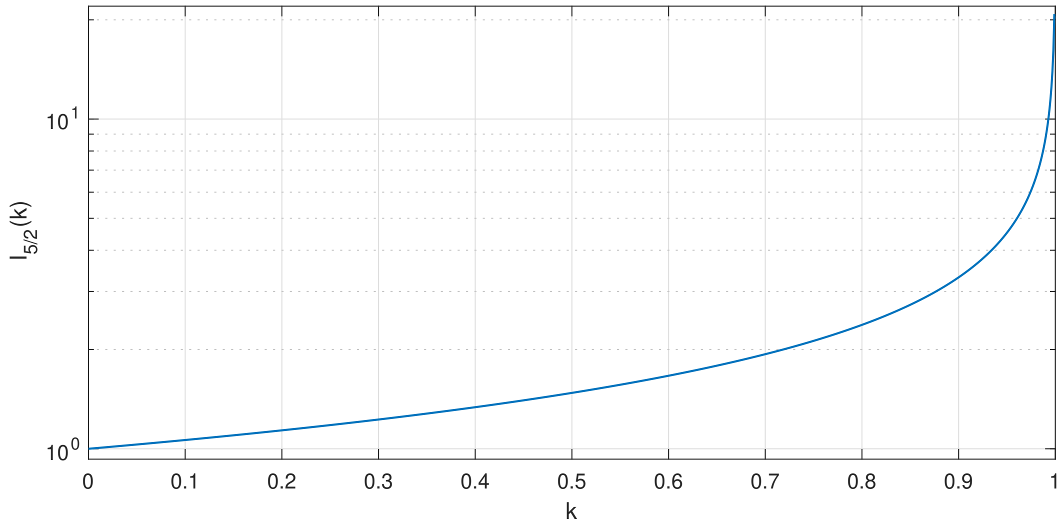
$$y \sim \frac{1}{\sqrt{\omega}}(a\cos\theta + b\sin\theta), \quad \omega^2 = f > 0, \quad \theta = \left|\int_{x_*}^x \omega(x')\, dx'\right|, \quad f(x_*) = 0,$$

$$y \sim \frac{1}{\sqrt{2\Omega}}\left[(a - b)e^{\Phi} + 2(a + b)e^{-\Phi}\right], \quad \Omega^2 = -f > 0, \quad \Phi = \left|\int_{x_*}^x \Omega(x')\, dx'\right|,$$

and

$$w'' + \Lambda^2 x^{p-2} w = 0,$$

has solution, $w(x) = \sqrt{x}\, \mathcal{C}_{1/p}(2\Lambda x^{p/2}/p)$, where $\mathcal{C}_\nu(z)$ is a Bessel function of order ν.



5. (a) Find the general term in an expansion for $k \ll 1$ of

$$I_a(k) = \int_0^1 \frac{\log(x^{-1})\, dx}{(1-kx)^a},$$

where a is a positive parameter.

(b) Using this expansion, find the nearest singularity to the origin k_0 and its type, α. Then make a second approximation of the integral to find its leading-order behaviour for $k \to k_0$.

(c) Use multiplicative and additive extraction to remove the nearest singularity in the small-k approximation, keeping terms upto and including order k^2.

(d) Construct the (2,2) Padé approximant from the $k \ll 1$ series solution. At what value of k does this approximant place the singularity for $a = 7/3$?

(e) Compute $S_n(k)$, the $(n+1)-$term approximation of $I_{7/3}(k)$ for $k \ll 1$ and $n = 0$ to 6. Sketch $S_6(k)$ against k for $0 \leq k \leq 1$, and compare the result with the numerical computation shown in the figure; **use a printout of the figure if needed!** Next use the Shanks transform to generate improved approximations of $I_{7/3}(k)$; iterate the transform to find even better approximations. Add the best of these to your figure, together with (for $a = 7/3$) the leading-order solution for $k \to k_0$, the two improved series from (c), and the (2,2) Padé approximant from (d). Finally, for a numerical comparison, compare all these results with the numerically determined value, $I_{7/3}(0.9) \approx 3.106$. Which is superior?