Asymptotics final

Answer as much as you can. May the force be with you.

1. For $z \gg 1$, use Laplace's method to find the leading-order approximation to the integral,

$$\int_0^\infty e^{t-z(t^4-2t^2)} \sin^2(2\pi\nu t)\, dt,$$

allowing for any value of the parameter $\nu > 0$.

2(a). Find the coefficients of the $\xi \ll 1$ asymptotic approximation,

$$I = \int_\xi^\infty e^{-\alpha t}\frac{dt}{t} \sim a_0\, \log\xi + a_1 + a_2\xi.$$

2(b). The function $y(x)$ satisfies the equation,

$$y_{xx} + \frac{2}{x}y_x = 2\epsilon(y^2-1)$$

in $x > 1$, with $\epsilon > 0$ and the boundary conditions, $y(1) = 0$ and $y \to 1$ as $x \to \infty$. Using the asymptotic sequence $(1, \epsilon^{1/2}, \epsilon\log\epsilon^{-1}, \epsilon)$, obtain an asymptotic expansion for the near-field solution, $y(x)$, with $x = O(1)$, Then find an equivalent far-field solution for $y = Y(X)$, with $X = \delta(\epsilon)x = O(1)$, for some $\delta(\epsilon) \ll 1$ that you should determine. Match the two solutions. *Note that the substitution $Y(X) = u(X)/X$ proves handy in finding homogeneous solutions to the ODE, $Y'' + \frac{2}{X}Y' - 4Y = F(X)$, and reduction of order or variation of constants should help with finding a particular solution.*

3. Using multiple scales, find the leading-order asymptotic approximation, valid for $t = O(\epsilon^{-1})$, to the solution of the equations,

$$\ddot{x} + x - y = \epsilon[x - \mathrm{Max}(\dot{x}, 0)], \qquad \dot{y} = \frac{1}{4}\epsilon\left(\dot{x}\cos t - y\right), \qquad x(0) = 1, \qquad \dot{x}(0) = 0, \qquad y(0) = 0.$$

4. Using the WKB method, provide an approximation for the eigenvalue, λ, of the problem

$$y'' + \pi^2\lambda y(1 + 2\cos\pi x)\sin^2(\pi x/2) = 0, \qquad 0 \le x \le 1, \qquad y(0) = y(1) = 0\ .$$

Compare your result with the first five eigenvalues obtained numerically: $|\lambda| \approx 3.18$, 8.62, 22.26, 48.64 and 58.62. *Note that the WKB approximation to $y'' + f(x)y = 0$ is*

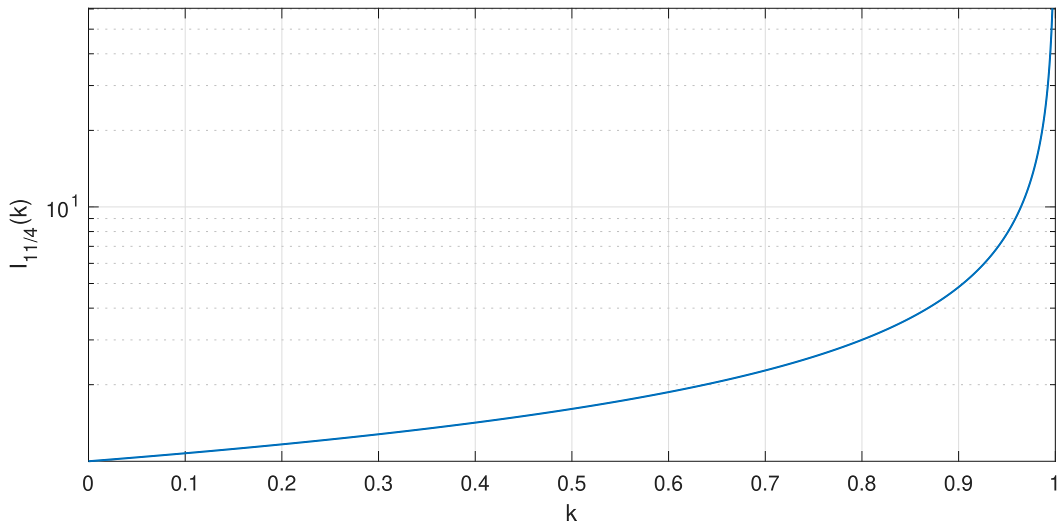
$$y \sim \frac{1}{\sqrt{\omega}}(a\cos\theta + b\sin\theta), \quad \omega^2 = f > 0, \quad \theta = \left|\int_{x_*}^x \omega(x')\,dx'\right|, \quad f(x_*) = 0,$$

$$y \sim \frac{1}{\sqrt{2\Omega}}\left[(a-b)e^{\Phi} + 2(a+b)e^{-\Phi}\right], \quad \Omega^2 = -f > 0, \quad \Phi = \left|\int_{x_*}^x \Omega(x')\,dx'\right|,$$

and

$$w'' + \Lambda^2 x^{p-2} w = 0,$$

has solution, $w(x) = \sqrt{x}\,\mathcal{C}_{1/p}(2\Lambda x^{p/2}/p)$, where $\mathcal{C}_\nu(z)$ is a Bessel function of order ν.



5. (a) Find the general term in an expansion for $k \ll 1$ of

$$I_a(k) = \int_0^1 \frac{\log(x^{-1})\, dx}{(1-kx)^a},$$

where a is a positive parameter.

(b) Using this expansion, find the nearest singularity to the origin k_0 and its type, α. Then make a second approximation of the integral to find its leading-order behaviour for $k \to k_0$.

(c) Use multiplicative and additive extraction to remove the nearest singularity in the small-k approximation, keeping terms upto and including order k^2.

(d) Construct the (2,2) Padé approximant from the $k \ll 1$ series solution. At what value of k does this approximant place the singularity for $a = 11/4$?

(e) Compute $S_n(k)$, the $(n+1)-$term approximation of $I_{11/4}(k)$ for $k \ll 1$ and $n = 0$ to 6. Sketch $S_6(k)$ against k for $0 \leq k \leq 1$, and compare the result with the numerical computation shown in the figure; **use a printout of the figure if needed!**

(f) Next use the Shanks transform to generate improved approximations of $I_{11/4}(k)$; iterate the transform to find even better approximations. Add the best of these to your figure, together with (for $a = 11/4$) the leading-order solution for $k \to k_0$, the two improved series from (c), and the (2,2) Padé approximant from (d).

(g) Finally, for a numerical comparison, compare all these results with the numerically determined value, $I_{11/4}(0.9) \approx 4.840$. Which is superior?