Asymptotics final

Answer as much as you can. The questions do not have equal weight. There are two pages.

1. For $z \gg 1$, use Laplace's method to find the leading-order approximation to the integral,

$$\int_0^\infty e^{-zf(t)} g(t)\, dt, \quad f(t) = 4t^5 - 5(3+a)t^4 + 20at^3, \quad g(t) > 0 \text{ for } t \ge 0,$$

where a is a parameter.

2(a). Find the coefficients of the $\rho \to 0$ asymptotic approximation to the integral, $\int_\rho^\infty t^{-3} e^{-t} dt$, noting that Euler's constant is $\gamma = -\int_0^\infty e^{-x} \log x\, dx$.

2(b). The function $f(r)$ satisfies the equation,

$$\frac{d^2 f}{dr^2} + \frac{3}{r}\frac{df}{dr} + \frac{1}{3}\epsilon(2+f)\frac{df}{dr} = 0,$$

in $r \ge 1$, with $\epsilon > 0$ and the boundary conditions, $f = 0$ on $r = 1$ and $f \to 1$ as $r \to \infty$. Obtain an asymptotic expansion for f at fixed r as $\epsilon \to 0$ in the asymptotic sequence, 1, ϵ, $\epsilon^2 \log(1/\epsilon)$, ϵ^2, ... Then find an expansion for f at fixed $\rho = \epsilon r$ as $\epsilon \to 0$ in the sequence, 1, ϵ^2, ... Match these expressions.

3. Using multiple scales, find the leading-order asymptotic approximation, valid for $t = O(\epsilon^{-1})$ to the solution of the equations,

$$\dot{x} = y + \epsilon x^2 z, \quad \dot{y} = -x + \epsilon y, \quad \dot{z} = x - \epsilon z, \qquad x(0) = 0, \quad y(0) = 1, \quad z(0) = 0.$$

4. Using the WKB method, provide an approximation for the eigenvalue, λ, of the problem

$$y'' + \pi^2 \lambda y (1 + 3\cos \pi x)\sin^2 \pi x = 0, \qquad 0 \le x \le 1, \qquad y(0) = y(1) = 0 \ .$$

Compare your result with the first five solutions obtained numerically: $|\lambda| \approx 1.06$, 7.91, 9.18, 21.16 and 40.67. *Note that the WKB approximation to $y'' + f(x)y = 0$ is*

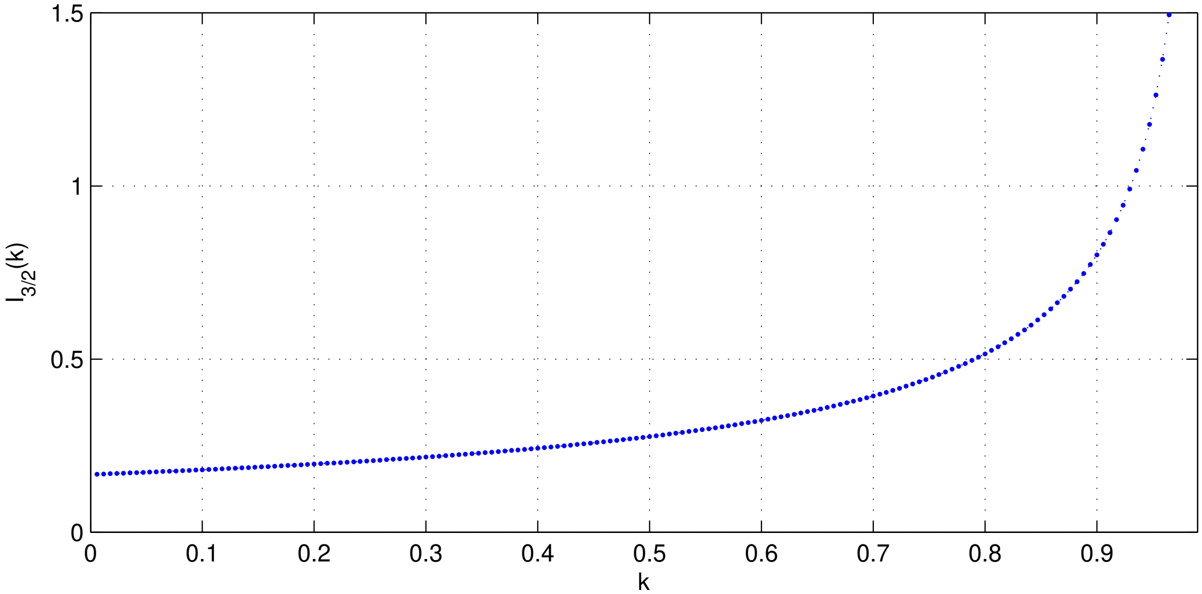
$$y \sim \frac{1}{\sqrt{\omega}}(a\cos\theta + b\sin\theta), \quad \omega^2 = f > 0, \quad \theta = \left| \int_{x_*}^x \omega(x')\, dx' \right|, \quad f(x_*) = 0,$$

$$y \sim \frac{1}{\sqrt{2\Omega}}\left[(a-b)e^{\Phi} + 2(a+b)e^{-\Phi}\right], \quad \Omega^2 = -f > 0, \quad \Phi = \left| \int_{x_*}^x \Omega(x')\, dx' \right|,$$

and

$$w'' + \Lambda^2 x^{p-2} w = 0,$$

has solution, $w(x) = \sqrt{x}\, \mathcal{C}_{1/p}(2\Lambda x^{p/2}/p)$, *where* $\mathcal{C}_\nu(z)$ *is a Bessel function of order* ν.



5. (a) For the integral,

$$I_\beta(k) = \int_0^1 \frac{x(1-x)\,dx}{(1-kx^2)^{5/2}},$$

find the general term in an expansion for $k \ll 1$.

(b) Next, from the $k \ll 1$ series solution find the nearest singularity to the origin k_0 and its type, α. Returning to the integral, make a second approximation about the singularity, obtaining the first two terms of the approximation $I(k) \sim J_0 + J_\alpha(k_0 - k)^\alpha$.

(c) Use multiplicative and additive extraction to remove the nearest singularity in the small-k approximation, keeping terms upto and including order k^2.

(d) For $n = 0$ to 7, compute $S_n(k)$, the $(n+1)-$term approximation of $I(k)$. Use the Shanks transform to generate five improved approximations of $I(k)$. Iterate the Shanks transform to find even better approximations.

(e) Construct the (2,2) Padé approximant from the $k \ll 1$ series solution. At what value of k does this approximant place the singularity?

For $0 \le k \le 1$, sketch $S_7(k)$ against k and compare the result with the numerical computation shown in the figure; **use a printout of the figure if needed!** To the plot, add $I(k) \sim J_0 + J_\alpha(k_0 - k)^\alpha$, the two improved series from (c), the best approximation from the Shanks transforms in (d), and the (2,2) Padé approximant from (e). For a numerical comparison, compare all these results with the numerically determined value, $I(0.9) \approx 0.8008$.