

Algebraic problems

1. Find the rescalings for the roots of

$$\epsilon^2 x^3 + (1 + 7\epsilon + 4\epsilon^2)x^2 + (12 + 16\epsilon + 18\epsilon^2)x + 35 + \epsilon + 9\epsilon^2 = 0,$$

and thence find two (non-trivial) terms in the approximation for each root, using (a) iteration and (b) expansion.

2. Develop two terms of the perturbation solutions to

$$\delta x^3 - (2 + 17\delta + 14\delta^2 + 14\delta^3)x^2 + (14 + 98\delta + 203\delta^2 + 283\delta^3 + 126\delta^4)x - 70\delta - 636\delta^2 - 1682\delta^3 - 1260\delta^4 = 0,$$

for $\delta \ll 1$ and $\delta \gg 1$.

3. Develop perturbation solutions to

$$x^3 - (63 + 36\epsilon + 18\epsilon^2)x^2 + (1323 + 1953\epsilon + 1140\epsilon^2 + 576\epsilon^3)x - 9261 - 25137\epsilon - 17766\epsilon^2 - 21058\epsilon^3 - 4608\epsilon^4 = 0$$

finding the three terms in the approximation for each root, $x = x_0 + \epsilon^\alpha x_\alpha + \epsilon^{2\alpha} x_{2\alpha}$, and determining α along the way.

4. Develop three terms of the perturbation solutions to the real roots of

$$e^{-x} \tanh(x - 2) = \epsilon,$$

identifying the scalings in the expansion sequence $\delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2 + \dots$

Eigenproblems and regularly perturbed differential equations

1. Find the corrections to the leading-order eigenvalues of the matrix problem

$$\begin{pmatrix} 2 & \alpha \\ \beta & 2 \end{pmatrix} \mathbf{x} = \lambda \mathbf{x} + \epsilon \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \mathbf{x},$$

for all possible values of the real parameters α and β .

2. Find the (nontrivial) nonlinear correction to the leading-order eigenvalues λ_0 of the differential equation,

$$y'' + \lambda y + \epsilon y^2 = 0, \quad y(0) = y(\pi) = 0, \quad \frac{2}{\pi} \int_0^\pi y^2 dx = 1.$$

3. Normal modes of a slightly mis-shapen membrane

Normal-mode solutions to the wave equation $\nabla^2 \phi = \phi_{tt}$ take the form $\phi(x, y, t) = \Phi(x, y) \cos(\omega t)$ and therefore satisfy

$$\Phi_{xx} + \Phi_{yy} = -\omega^2 \Phi,$$

where subscripts denote partial derivatives. Consider a slightly mis-shapen membrane covering the domain,

$$0 \leq x \leq \pi, \quad \epsilon x(\pi - x) \leq y \leq \pi,$$

with $\Phi = 0$ on the boundary. Show that $\Phi = \sin nx \sin my$ is a leading-order eigenfunction, with n and m integers. Find the corresponding eigenvalue ω . Calculate the $O(\epsilon)$ correction to the eigenvalue for (a) $(n, m) = (1, 1)$ and (b) $(n, m) = (2, 1)$.

Integrals

1. Use the method of repeated integration by parts or rescaling to obtain five terms in the asymptotic approximation to the integral,

$$\int_x^\infty t^{-5} e^{-t^2} dt,$$

for $x \rightarrow 0$. Note that

$$\gamma = - \int_0^\infty e^{-u} \log u \, du$$

is Euler's constant.

2. Find the leading-order behaviour for $x \gg 1$ of

$$(a) \int_0^\infty e^{xt^3(5-3t^2)} \frac{t^2 dt}{(1+t^2)} \quad (b) \int_{-\pi}^\pi t^2 e^{-x \sin^2 t} dt \quad (c) \int_0^\pi \sqrt{\sinh t} e^{-x \sinh^5 t} dt$$

3. Evaluate the first two terms as $\epsilon \rightarrow 0$ of

$$\int_0^\infty \frac{dx}{(\epsilon + x)^{j/2} (1+x)}$$

for $j = 1, 2, 3$.

4. Evaluate the first two terms as m approaches unity from below of

$$\int_{-\pi/4}^{\pi/4} (1 - m^2 \tan^2 \theta)^{-1/2} d\theta$$