Asymptotics final

Answer as much as you can. The questions do not have equal weight. There are two pages.

1. For $z \gg 1$, find the leading-order approximation to the integral,

$$I_j = \int_{-j}^{j} g(t) e^{-zf(t)} dt, \qquad 1 < g(t) < 2, \qquad f(t) = t^5 + 7t^4 + 10t^3 - 18t^2 - 27t + 27,$$

where j is any integer.

2(a). Find the coefficients of the $\rho \to 0$ asymptotic approximation to the integral, $\int_\rho^\infty t^{-2} e^{-4t} dt$, noting that Euler's constant is $\gamma = -\int_0^\infty e^{-x} \log x \, dx$.

2(b). The function $f(r)$ satisfies

$$f_{rr} + \frac{2}{r} f_r + 4\epsilon^2 r f^2 f_r = 0, \qquad \text{in } r > 1,$$

and is subject to the boundary conditions, $f(1) = 0$ and $f \to 1$ as $r \to \infty$. Obtain an asymptotic expansion for f at fixed r and $\epsilon \to 0$ in the asymptotic sequence, $\{1, \epsilon, \epsilon^2 \log(1/\epsilon), \epsilon^2\}$. Then find a similar expansion for f at fixed $\rho = \epsilon^\alpha r$, where $\alpha > 0$ is a parameter that you should determine. Match these expansions.

3. Using multiple scales, find the leading-order asymptotic approximation, valid for $t = O(\epsilon^{-1})$ to the solution of the equations,

$$\dot{x} = y, \qquad \dot{y} + x = z + \epsilon z y^m, \qquad \dot{z} = \epsilon(xy + yz - z), \qquad y(0) = 0, \quad x(0) = z(0) = 1,$$

where m is a positive integer.

4. Using the WKB method, provide approximations for **all** eigenvalues, λ, of the problem

$$y'' + \lambda y F(x) = 0, \qquad y(0) = y(2) = 0, ,$$

where $F(x) \to cx^3$ for $x \to 0$, with c a positive constant, $F(1) = 0$, $F > 0$ for $0 < x < 1$, and $F < 0$ for $1 < x < 2$. *Note that the WKB approximation to* $y'' + f(x)y = 0$ *is*

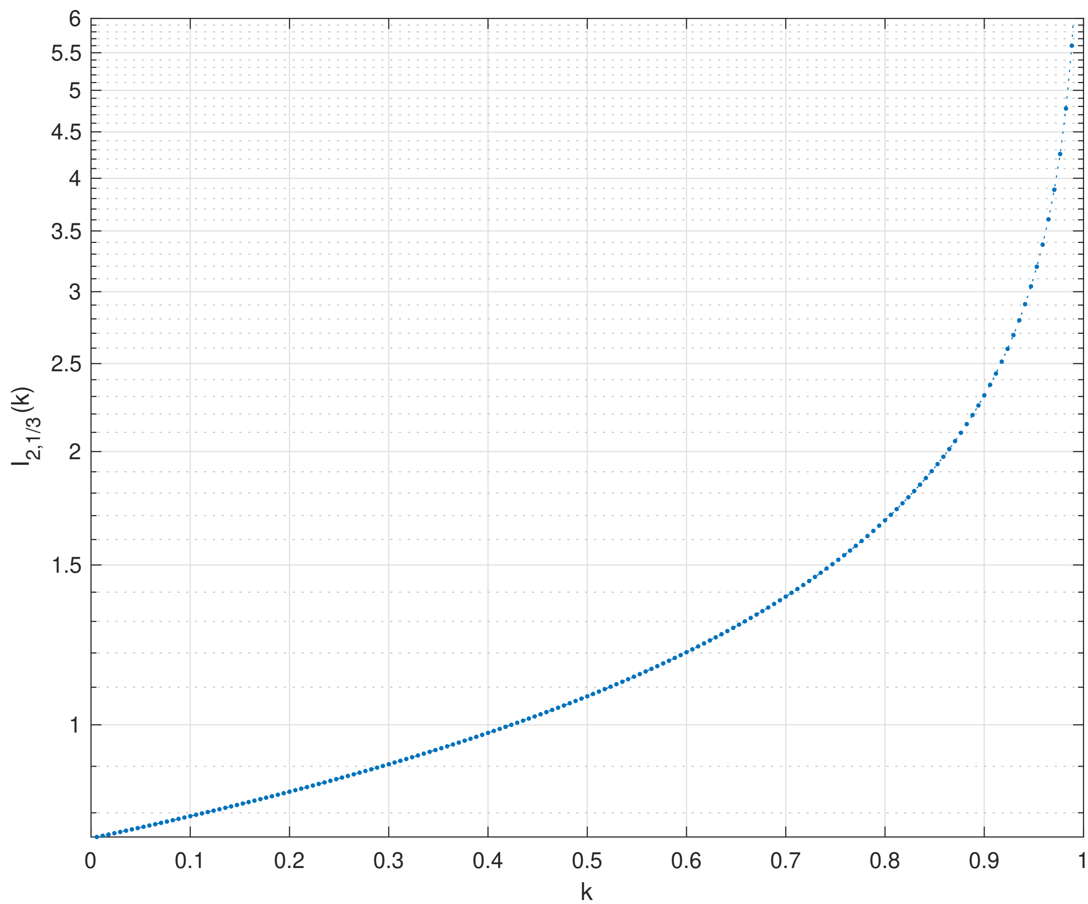
$$y \sim \frac{1}{\sqrt{\omega}}(a\cos\theta + b\sin\theta), \quad \omega^2 = f > 0, \quad \theta = \left| \int_{x_*}^{x} \omega(x')\, dx' \right|, \quad f(x_*) = 0,$$

$$y \sim \frac{1}{\sqrt{2\Omega}} \left[(a-b)e^{\Phi} + 2(a+b)e^{-\Phi} \right], \quad \Omega^2 = -f > 0, \quad \Phi = \left| \int_{x_*}^{x} \Omega(x')\, dx' \right|,$$

and

$$w'' + z^{p-2} w = 0,$$

has solution, $w(z) = \sqrt{z}\, \mathcal{C}_{1/p}(2z^{p/2}/p)$*, where* $\mathcal{C}_\nu(x)$ *is a Bessel function of order* ν*.*



5. (a) With positive parameters β and γ, find the general term in an expansion for $k \ll 1$ of

$$I_{\beta,\gamma}(k) = \int_0^1 \frac{x^\gamma dx}{(1 - kx^\beta)^{1+\gamma}}$$

(b) Using this expansion, find the nearest singularity to the origin k_0 and its type, α. Then make a second approximation of the integral to obtain the coefficient A in $I_{\beta,\gamma}(k) \sim A(k_0 - k)^\alpha$.

(c) Use multiplicative and additive extraction to remove the nearest singularity in the small-k approximation, keeping terms upto and including order k^2.

(d) Construct the (2,2) Padé approximant from the $k \ll 1$ series solution. At what value of k does this approximant place the singularity when $\beta = 2$ and $\gamma = 1/3$?

(e) Compute $S_n(k)$, the $(n+1)-$term approximation of $I_{2,1/3}(k)$ for $k \ll 1$ and $n = 0$ to 7. Sketch $S_7(k)$ against k for $0 \leq k \leq 1$, and compare the result with the numerical computation shown in the figure; **use a printout of the figure if needed!** Next use the Shanks transform to generate improved approximations of $I_{2,1/3}(k)$. Add the best of these to your figure, together with $I_{2,1/3}(k) \sim A(k_0 - k)^\alpha$, the two improved series from (c), and the (2,2) Padé approximant from (d).

(f) For a numerical comparison, compare all the preceding results with the numerically determined value, $I_{2,1/3}(0.9) \approx 2.3060$.