

Algebraic problems

1. Find the rescalings for the roots of

$$\epsilon^4 x^3 + (1 + 7\epsilon - 6\epsilon^2)x^2 + (1 + 15\epsilon + 21\epsilon^2)x - 30 + 8\epsilon + 13\epsilon^2 = 0,$$

and thence find two (non-trivial) terms in the approximation for each root, using (a) iteration and (b) expansion.

2. Develop two terms of the perturbation solutions to

$$\delta x^3 - (9\delta + 5\delta^2 + 16\delta^3)x^2 + (18\delta + 66\delta^2 + 60\delta^3 - 112\delta^4)x - 216\delta^2 + 216\delta^3 + 672\delta^4 = 0,$$

for $\delta \ll 1$ and $\delta \gg 1$.

3. Develop perturbation solutions to

$$x^3 - (57 + \epsilon + 19\epsilon^2)x^2 + (1083 + 399\epsilon + 706\epsilon^2 - 76\epsilon^3)x - 6859 - 7220\epsilon - 8360\epsilon^2 - 5435\epsilon^3 - 76\epsilon^4 = 0$$

finding the three terms in the approximation for each root, $x = x_0 + \epsilon^\alpha x_\alpha + \epsilon^{2\alpha} x_{2\alpha}$, and determining α along the way.

4. Develop three terms of the perturbation solutions to the real roots of

$$xe^{-x^3} = 2\epsilon,$$

identifying the scalings in the expansion sequence $\delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2 + \dots$

Eigenproblems and regularly perturbed differential equations

1. Find the corrections to the leading-order eigenvalues of the matrix problem

$$\begin{pmatrix} 1 & \alpha \\ \beta & -1 \end{pmatrix} \mathbf{x} = \lambda \mathbf{x} + \epsilon \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x},$$

for all possible values of the real parameters α and β .

2. Find the (nontrivial) nonlinear correction to the leading-order eigenvalues λ_0 of the differential equation,

$$y'' + \lambda y + \epsilon y^2 = 0, \quad y(0) = y(1) = 0, \quad \int_0^1 y^2 dx = \frac{1}{2}.$$

3. Normal modes of a slightly mis-shapen membrane

Normal-mode solutions to the wave equation $\nabla^2 \phi = \phi_{tt}$ take the form $\phi(x, y, t) = \Phi(x, y) \cos(\omega t)$ and therefore satisfy

$$\Phi_{xx} + \Phi_{yy} = -\omega^2 \Phi,$$

where subscripts denote partial derivatives. Consider a slightly mis-shapen membrane covering the domain,

$$-\pi \leq x \leq \pi, \quad -\pi + \epsilon f(x) \leq y \leq \pi,$$

with $\Phi = 0$ on the boundary, where $f(x)$ is a given function. Show that $\Phi = \sin nx \sin my$ is a leading-order eigenfunction, with n and m integers. Find the corresponding eigenvalue ω . Calculate the $O(\epsilon)$ correction to the eigenvalue for (a) $(n, m) = (1, 1)$ and (b) $(n, m) = (2, 1)$. If $f(x)$ is an odd function, briefly describe what you would do to find the first non-trivial correction to the leading-order eigenvalue.