MATH 400 – Final exam

Closed book exam; no calculators. Answer as much as you can; credit awarded for the best three answers. The questions all have equal weight. Adequately explain the steps you take. *e.g.* if you use an expansion formula, say in one sentence why this is possible; if you quote a special function solution to an ODE, say why this is the correct one. Be as explicit as possible in giving your solutions.

1. Using separation of variables, find the solution to the PDE,

$$u_t = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right),$$

where $0 \le \theta \le \pi$, demanding that the solution be regular at $\theta = 0$ and π and satisfy the initial condition, $u(\theta, 0) = f(\theta)$. Find explicit solutions (involving no integrals) for $f(\theta) = \sin^2\theta$ and $f(\theta) = sign(x)$, where $x = \cos\theta$. For either case, does u remain non-zero for $t \to \infty$?

2. Establish that

$$f \circ g = \mathcal{F}^{-1}\{\hat{f}\hat{g}\}, \qquad \mathcal{F}^{-1}\{e^{-ika}\hat{f}(k)\} = f(x-a) \qquad \text{and} \qquad \mathcal{F}^{-1}\{e^{-k^2t}\} = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$$

where $\mathcal{F}\{f\} = \hat{f}(k)$, $\mathcal{F}\{g\} = \hat{g}(k)$ and $f \circ g$ is a convolution.

Consider the PDE

$$u_t + cu_x = u_{xx} + \alpha u \qquad u(x, 0) = f(x),$$

which models heat diffusion along a wire that is heated everywhere with rate α, and along which a wind blows with speed c. Use the Fourier transform to solve the PDE, expressing your solution in terms of a single integral. Give an explicit solution if $f(x) = \delta(x)$ (Dirac's delta function), showing that, at large times $t \gg 1$:
(i) the solution always grows exponentially in time along the path $x = ct$.
(ii) for fixed position x, the solution decays exponentially in time when c exceeds a threshold value that you should determine.
Interpret these results for the model problem.

3. Establish the relations,

$$\mathcal{L}\{f'(t)\} = s\overline{f}(s) - f(0) \qquad \text{and} \qquad \mathcal{L}\{t^n e^{-t}\} = n!/(s+1)^{n+1}$$

for the Laplace transform, where $\overline{f}(s) = \mathcal{L}\{f(t)\}$ and n is a positive integer or zero.

An age-structured model of a population is based on the PDE,

$$u_t + u_a = -\mu(a)u, \qquad 0 \le a, t < \infty, \qquad \mu(a) = \tanh(a),$$

where $u(a, t)$ dictates the number of individuals with age a at time t. Initially $u(a, 0) = 0$. Birth occurs via a creation event and reproduction at a particular age A:

$$u(0, t) = \frac{t^n e^{-t}}{n!} + u(A, t).$$

Find the Laplace transform of the population structure function, $\overline{u}(a, s)$. Using this function, compute the total number of individuals of all ages over all time, $\int_0^\infty \int_0^\infty u(a, t)dadt$, and hence determine if the population becomes extinct.

Bonus: If you really want to impress me, for $n = 0$, invert the Laplace transform to provide a solution for $u(a, t)$ in terms of an infinite series. It might help to argue that $\overline{u}(a, s)$ is analytic in s but for an infinite number of simple poles, being careful about the two cases $t + A > a$ and $t + A < a$.

4. Solve the PDE

$$u_t + uu_x = 0, \qquad u(x,0) = f(x) = \begin{cases} 0, & x < -1, \\ 1+x, & -1 \le x < 0, \\ 1-x, & 0 \le x < 1, \\ 0, & 1 \le x, \end{cases}$$

and show that a shock forms after a sufficient time, determining when and where this discontinuity appears. Draw the characteristic curves on a space-time diagram. Sketch the solution for u upto and beyond the formation of the shock, indicating how one can avoid a multivalued solution using an "equal-areas rule". Briefly justify that rule using the integral form of the conservation law corresponding to the PDE and derive a formula for the speed of a shock. Calculate the shock position and add its path to your characteristics diagram.

Helpful information:

The Sturm-Liouville differential equation:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda\sigma(x)y = 0.$$

Legendre's equation, with regular solution $y = P_n(x)$ at $x = \pm 1$, is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0; \qquad \int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}.$$

Bessel's equation is

$$z^2 y'' + zy' + (z^2 - m^2)y = 0,$$

and has the solution, $y = J_m(z)$, which is regular at $z = 0$.

Fourier Transform:

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad \& \qquad f(x) = \mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

Laplace Transform:

$$\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st}dt \qquad \& \qquad f(t) = \mathcal{L}^{-1}\{\overline{f}(s)\} = \frac{1}{2\pi i}\int_{C} \overline{f}(s)e^{st}ds,$$

where C is the Bromwich contour

Cauchy's theorem: if $F(z)$ has a simple pole at $z = z_*$, but is otherwise analytic inside a closed contour $\mathcal{C}$,

$$\int_{\mathcal{C}} F(z)dz = 2\pi i \left[(z - z_*)F(z)\right]_{z \to z_*}.$$

Convolution:

$$f \circ g = \int_{-\infty}^{\infty} f(x')g(x - x')dx'.$$

Helpful trigonometric relations:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \qquad \sin(A+B) = \sin A\cos B + \cos A\sin B.$$

MATH 400 – Solution

1. Let $u = Y(x)T(t)$ where $x = \cos\theta$. Then, separating variables,

$$T' + \lambda T = 0, \qquad [(1-x^2)Y']' + \lambda Y = 0.$$

Thus, T is given by $e^{-\lambda t}$ and Y satisfies Legendre's equation. *i.e.* Y is given by $P_n(x)$, with $\lambda = n(n+1)$ and $n = 0, 1, 2, ...$, on demanding regularity at $x = \pm 1$. We write a general solution,

$$u = \sum_{n=0}^{\infty} c_n e^{-n(n+1)t} P_n(x).$$

The coefficients c_{nj} must be chosen to fit the initial condition:

$$f(\theta) = u(\theta, 0) = \sum_{n=0}^{\infty} c_n P_n(x) \qquad \longrightarrow \qquad c_n = \frac{\int_{-1}^{1} f P_n dx}{\int_{-1}^{1} P_n^2 dx} \equiv (n + \tfrac{1}{2}) \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta d\theta,$$

in view of the expansion theorem of a Sturm-Liouville problem (of which Legendre's equation is an example, with a weight function of unity) and the helpful integral provided. *(5 points, some indication needed for choosing P_n and where the formula for c_n comes from.)*

For $f(\theta) = \sin^2\theta$, we have the alternative representation $f = 1 - x^2 = 2[P_0(x) - P_2(x)]/3$, given that $P_0 = 1$ and $P_2 = (3x^2 - 1)/2$ (which can be quoted from memory or computed using either Legendre's equation or the orthogonality relation, along with $P_n(1) = 1$). Hence, $u = [2P_0 - P_x(x)e^{-6t}]/3$. Evidently, $u \to 2/3$ for $t \to \infty$. *(3 points, some indication needed for where the polynomials come from.)*

For the odd function $f(\theta) = sign(\cos\theta)$, c_0 and all the even coefficients vanish, whereas the odd coefficients are

$$c_n = (2n+1) \int_0^1 P_n(x) dx = \frac{(2n+1)}{n(n+1)} P_n'(0), \quad n \neq 0,$$

in view of Legendre's equation. Because $c_0 = 0$, $u \to 0$ for $t \to \infty$. *(3 points.)*

2. From the definition of the Fourier transform and its inverse,

$$\mathcal{F}\{f \circ g\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ikx} g(x - x') f(x') dx dx' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ikz - ikx'} g(z) f(x') dx dz,$$

$$\mathcal{F}^{-1}\{e^{-ika} \hat{f}(k)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-a)} \hat{f}(k) dk \equiv f(x - a)$$

and

$$\mathcal{F}^{-1}\{e^{-k^2 t}\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - k^2 t} dk = \frac{1}{2\pi} e^{-x^2/4t} \int_{iZ-\infty}^{iZ+\infty} e^{-tz^2} dz,$$

where $Z = -x/(2t)$. By deforming the final integral back to the real axis, using Cauchy's theorem and the analyticity of the integrand, and then using $\int_{-\infty}^{\infty} e^{-tx^2} dx = \sqrt{\pi/t}$, we thereby establish all the desired results. *(4 points.)*

Transforming the PDE and initial condition:

$$\hat{u}_t = (\alpha - ikc - k^2)\hat{u} \qquad \longrightarrow \qquad \hat{u}(k,t) = \hat{f}(k) e^{(\alpha - ikc - k^2)t}.$$

Hence, using the results above,

$$u(x,t) = f(x) \circ G(x,t), \qquad G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp[\alpha t - (x - ct)^2/4t]. \qquad (3 \textit{ points})$$

For $f(x) = \delta(x)$, we obtain

$$u(x,t) = G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp[(\alpha - c^2/4)t - x^2/(4t) + cx/2].$$

Along the path $x = ct$, $u = e^{\alpha t}/\sqrt{4\pi t}$, whereas $u \propto e^{(\alpha - c^2/4)t}$ for any fixed x, and thus decay exponentially when $c^2 > 4\alpha$. Moving along with the wind, one experiences the heating of the wire, and the temperature grows exponentially; but at fixed position, if the wind blows too hard, the initial temperature distribution is swept by and the wire eventually cools down. *(4 points.)*

3. From the definitions (and as long as $\mathrm{Re}(s) > 0$ and $f(t)$ is bounded for $t \to \infty$),

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt = -f(0) - s\int_0^\infty f(t)e^{-st}dt = s\overline{f}(s) - f(0)$$

and

$$\mathcal{L}\{t^n e^{-t}\} = \int_a^\infty t^n e^{-st-t}dt = \frac{n}{(s+1)}\int_a^\infty t^{n-1}e^{-st-t}dt = \frac{n!}{(s+1)^{n+1}},$$

after integrating by parts as many times as needed. *(2 points.)*

For $u(a,0) = 0$, Laplace transforming the PDE gives

$$\bar{u}_a = -[s + \mu(a)]\bar{u} \qquad \longrightarrow \qquad \bar{u}(a,s) = \bar{u}(0,s)e^{-m(a)-sa}, \quad m(a) = \ln[\cosh(a)].$$

Taking the transform of the birth condition:

$$\bar{u}(0,s) = (s+1)^{-n-1} + \bar{u}(A,s) \qquad \longrightarrow \qquad \bar{u}(a,s) = \frac{e^{-m(a)-sa}}{(s+1)^{n+1}(1 - e^{-M-sA})}.$$

where $M = m(A)$. In view of the definition of the Laplace transform, the double integral of $u(a,t)$ is equal to

$$\int_0^\infty \bar{u}(a,0)da = \frac{1}{1 - e^{-M}}\int_0^\infty e^{-m(a)}da = \frac{1}{1 - \mathrm{sech}(M)}\int_0^\infty \mathrm{sech}(a)da = \frac{\frac{1}{2}\pi}{1 - \mathrm{sech}(M)}.$$

This is finite, implying that $\int_0^\infty u(a,t)da \to 0$ for $t \to \infty$. Hence the total number of individuals of any age eventually must decay to zero, implying extinction. *(9 points.)*

Bonus: With $n = 0$, the solution for $\bar{u}(a,s)$ depends on s through the analytic exponential function, and the two factors in the denominator of $s+1$ and $1 - e^{-M-sA}$. These factors vanish for $s = -1$ and $s = (-M + 2ij\pi)A^{-1}$ with $j = 0, \pm1, \pm2, ...$, implying an infinite set of simple poles. Thus, for the inverse Laplace transform, we may deform the Bromwich contour leftwards across the complex $s-$plane to encircle the poles if $t > a - A$ (as demanded by the limiting exponential factor $e^{s(t+A-a)}$). If $t < a - A$, the poles are not encircled by the righward deformation of the Bromwich contour. Cauchy's residue theorem then indicates that

$$u(a,t) = \left[\frac{e^{-m(a)-t+a}}{1 - e^{A-M}} + e^{-m(a)+(M/A)(a-t)}\sum_{j=-\infty}^{\infty}\frac{Ae^{2ij\pi(t-a)/A}}{A - M + 2ij\pi}\right]H(t + A - a).$$

(3 points).

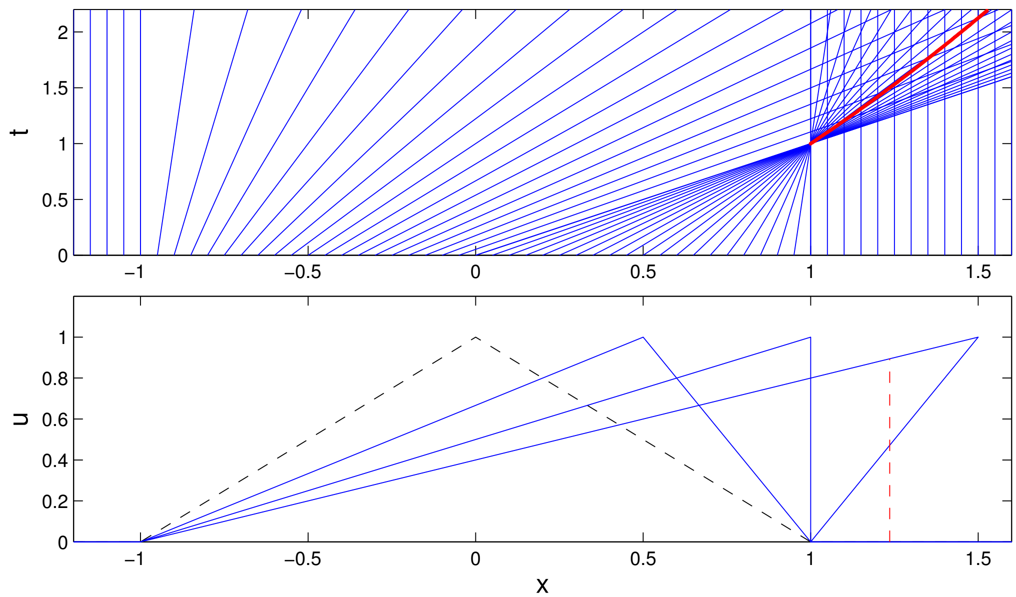
4. The characteristics equations and solution:

$$\frac{dx}{dt} = u \quad \& \quad \frac{du}{dt} = 0 \qquad \longrightarrow \qquad x = x_0 + ut \quad \& \quad u = f(x + ut).$$

Given the piece-wise linear form of the initial condition,

$$u = \begin{cases} 0, & x < -1, \\ (1+x)/(1+t), & -1 \le x < t, \\ (1-x)/(1-t), & t \le x < 1, \\ 0, & 1 \le x, \end{cases}$$

which breaks down for $t = 1$ at $x = 1$ when the third region shrinks to xero width and the solution there becomes vertical. *(4 points).*



The integral form of the conservation law is

$$\frac{d}{dt}\int_a^b u(x,t)dx = -\frac{1}{2}[u^2(x,t)]_a^b$$

For the equal-areas rule, one surgically removes the multivalued part of the solution by introducing a vertical line that cuts out equal area to either side; this is justified by the integral form of the conservation law which demands that $\int_{-\infty}^{\infty} u dx$ (the signed area underneath the curve of u) is constant in time if there is no flux into or out of the full spatial domain ($a \to -\infty$, $b \to \infty$). If u jumps from u^- to u^+ at $x = X(t)$, then

$$\frac{d}{dt}\int_a^X u(x,t)dx + \frac{d}{dt}\int_X^b u(x,t)dx = \int_a^X u_t(x,t)dx + \int_X^b u_t(x,t)dx - (u^+ - u_-)\frac{dX}{dt} = -\frac{1}{2}[u^2(x,t)]_a^b$$

Taking the limits $a \to X^-$ and $b \to X^+$ now gives

$$\frac{dX}{dt} = \frac{1}{2}(u^+ + u^-).$$

For the problem at hand, we have $u^- = (X+1)/(t+1)$ and $u^+ = 0$, which gives $\dot{X} = \frac{1}{2}(X+1)/(t+1)$ and $X = \sqrt{2(t+1)} - 1$ (given the shock's initial position). *(4 points).*

Sketches: 3 points.