MATH 400 – Final exam

Closed book exam; no calculators. Answer as much as you can; credit awarded for the best three answers. Adequately explain the steps you take. *e.g.* if you use an expansion formula, say in one sentence why this is possible; if you quote a special function solution to an ODE, say why this is the correct one. Be as explicit as possible in giving your solutions.

1. The ringing of a hemispherical bell is modelled by

$$u_{tt} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right), \qquad 0 \le \theta \le \tfrac{1}{2}\pi.$$

Using separation of variables, find the solution, demanding regularity at $\theta = 0$, and imposing the boundary and initial conditions, $u(\frac{1}{2}\pi, t) = u(\theta, 0) = 0$ and $u_t(\theta, 0) = f(\theta)$. What are the normal-mode frequencies (*i.e.* the possible choices for ω if $u \propto \sin\omega t$)? Find an explicit solution (involving no integrals) for $f(\theta) = 1 + 5\cos^3\theta - 3\cos\theta$.

2. Establish that

$$\mathcal{F}\{xf(x)\} = i\frac{d\hat{f}}{dk},$$

where $\mathcal{F}\{f\} = \hat{f}(k)$ is the Fourier transform of $f(x)$. Next consider the Airy function $\mathrm{Ai}(x)$, which satisfies

$$\frac{d^2\mathrm{Ai}}{dx^2} = x\mathrm{Ai}, \qquad \int_{-\infty}^{\infty} \mathrm{Ai}(x)\,dx = 1, \qquad \mathrm{Ai} \to 0 \text{ for } x \to \pm\infty.$$

By applying the Fourier transform, show that

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(\tfrac{1}{3}t^3 + xt)\,dt.$$

Now solve the PDE

$$u_t = u_x - u_{xxx}, \qquad u \to 0 \text{ for } x \to \pm\infty, \qquad u(x,0) = \mathrm{Ai}(x),$$

expressing your answer in terms of the Airy function (without any further integrals).

3. For the Laplace transform, establish the relations,

$$\mathcal{L}\{f'(t)\} = s\overline{f}(s) - f(0), \qquad \mathcal{L}\{f(t-a)H(t-a)\} = e^{-sa}\overline{f}(s) \qquad \text{and} \qquad \mathcal{L}\{\delta(t-a)\} = e^{-sa}$$

where $\overline{f}(s) = \mathcal{L}\{f(t)\}$, $a > 0$, $H(x)$ is the step function and $\delta(x)$ is the Dirac delta function.

An age-structured model of a population with larvae and adults is based on the PDEs,

$$u_t + u_a + \mu u = v, \qquad v_t + v_a + \mu v = -v, \qquad 0 \le a, t < \infty,$$

where μ is the (constant) death rate. Initially $u(a,0) = \delta(a - A)$ and $v(a,0) = 0$, for some age $A > 0$. The adults reproduce to create larvae such that

$$v(0,t) = \frac{3}{4}\int_0^\infty u(a,t)da,$$

whereas $u(0,t) = 0$. By Laplace transforming the equations, find $u(a,t)$ and $v(a,t)$.

4. Using the method of characteristics, solve the PDE,

$$u_t + u^2 u_x = 0, \qquad u(x,0) = f(x).$$

For the initial condition,

$$f(x) = \begin{cases} 0, & x < -1, \\ 1, & -1 \le x < 0, \\ \sqrt{1-x}, & 0 \le x < 1, \\ 0, & 1 \le x, \end{cases}$$

show that an expansion fan is launched from $x = -1$, and that a shock forms at $x = t = 1$. Sketch the characteristic curves on a space-time diagram, and snapshots of the solution for $t = \frac{1}{2}$, $t = \frac{3}{2}$ and $t = 3$, indicating how one can use a geometrical construction to avoid a multivalued solution for $t > 1$. Provide an equation of motion for the position of the shock, and then solve it to find the path of that discontinuity for $1 < t < \frac{5}{2}$ and $t > \frac{5}{2}$. Add the shock path to your characteristics diagram.

Helpful information:

The Sturm-Liouville differential equation:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda\sigma(x)y = 0.$$

Legendre's equation, with regular solution $y = P_n(x)$ at $x = \pm 1$, is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0; \qquad \int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}.$$

Bessel's equation is

$$z^2 y'' + zy' + (z^2 - m^2)y = 0,$$

and has the solution, $y = J_m(z)$, which is regular at $z = 0$.

Fourier Transform:

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad \& \qquad f(x) = \mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

Laplace Transform:

$$\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st}dt \qquad \& \qquad f(t) = \mathcal{L}^{-1}\{\overline{f}(s)\} = \frac{1}{2\pi i}\int_{C} \overline{f}(s)e^{st}ds,$$

where C is the Bromwich contour

Cauchy's theorem: if $F(z)$ has a simple pole at $z = z_*$, but is otherwise analytic inside a closed contour $\mathcal{C}$,

$$\int_{\mathcal{C}} F(z)dz = 2\pi i\left[(z - z_*)F(z)\right]_{z \to z_*}.$$

Convolution:

$$f \circ g = \int_{-\infty}^{\infty} f(x')g(x-x')dx'.$$

Helpful trigonometric relations:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \qquad \sin(A+B) = \sin A\cos B + \cos A\sin B.$$

MATH 400 – Solution

1. Let $u = Y(x)T(t)$ where $x = \cos\theta$. Then, separating variables,

$$T'' + \lambda T = 0, \qquad [(1-x^2)Y']' + \lambda Y = 0.$$

Thus, T is given by $\cos\omega t$ or $\sin\omega t$, with $\omega^2 = \lambda$, and Y satisfies Legendre's equation. The boundary conditions are unusual: X is regular at $x = 1$ and $X(0) = 0$. The problem still has Sturm-Liouville form, and the solutions are the *odd* Legendre polynomials $Y = P_n(x)$, with $\lambda = n(n+1)$ and $n = 1$, 3, 5, ... We write a general solution,

$$u = \sum_{n=1, n\ odd}^{\infty} (a_n \cos\omega_n t + b_n \sin\omega_n t) P_n(x) \qquad \omega_n = \sqrt{n(n+1)}.$$

But $a_n = 0$ in view of $u(\theta, 0) = 0$, and

$$f(\theta) = \sum_{n=1, n\ odd}^{\infty} \omega_n b_n P_n(x) \qquad \longrightarrow \qquad b_n = \frac{\int_0^1 f P_n dx}{\omega_n \int_0^1 P_n^2 dx} \equiv \frac{(2n+1)}{\sqrt{n(n+1)}} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta d\theta,$$

in view of the expansion theorem of a Sturm-Liouville problem and the helpful integral provided. *(7 points.)*

For $f(\theta) = 1 + 5\cos^3\theta - 3\cos\theta = 1 + 2P_3(x)$, we have

$$b_n = \frac{(2n+1)}{\sqrt{n(n+1)}} \int_0^1 (1 + 2P_3) P_n dx = \frac{(2n+1)}{\sqrt{n(n+1)}} \left[\frac{P_n'(0)}{n(n+1)} + \frac{2}{7}\delta_{n3} \right]$$

in view of Legendre's equation and the orthogonality of the P_n's. *(4 points.)*

2. From the definition of the Fourier transform

$$\mathcal{F}\{xf(x)\} = \int_{-\infty}^{\infty} e^{-ikx} x f(x) dx = i\frac{d}{dk} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \equiv i\frac{d\hat{f}}{dk} \qquad (1 \textit{ point}).$$

Transforming the ODE:

$$\frac{d}{dk}\widehat{\mathrm{Ai}} = ik^2 \widehat{\mathrm{Ai}} \qquad \longrightarrow \qquad \widehat{\mathrm{Ai}}(k) = \widehat{\mathrm{Ai}}(0) e^{ik^3/3}.$$

But

$$\widehat{\mathrm{Ai}}(0) = \int_{-\infty}^{\infty} \mathrm{Ai}(x)\ dx = 1.$$

Hence,

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx + ik^3/3} dk = \frac{1}{\pi} \int_0^{\infty} \cos(kx + \tfrac{1}{3}k^3) dk \qquad (4 \textit{ points}).$$

Last, Fourier transforming the PDE and initial condition,

$$\hat{u}_t = ik\hat{u} + ik^3\hat{u} \qquad \longrightarrow \qquad \hat{u}(k,t) = \hat{u}(k,0) e^{i(k^3+k)t} = e^{i(k^3+k)t + ik^3/3}.$$

Hence, inverting the transform,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x+t) + ik^3(1+3t)/3} dk = (1+3t)^{-1/3} \mathrm{Ai}\left(\frac{x+t}{(1+3t)^{1/3}} \right). \qquad (4 \textit{ points}).$$

3. From the definitions (and as long as $\mathrm{Re}(s) > 0$ and $f(t)$ is bounded for $t \to \infty$),

$$\mathcal{L}\{f'(t)\} = \int_0^{\infty} f'(t) e^{-st} dt = -f(0) - s\int_0^{\infty} f(t) e^{-st} dt = s\overline{f}(s) - f(0)$$

$$\mathcal{L}\{f(t-a)H(t-a)\} = \int_a^\infty f(t-a)e^{-st}dt = \int_0^\infty f(\tau)e^{-s\tau-sa}dt = e^{-sa}\overline{f}(s)$$

and

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty \delta(t-a)e^{-st}dt = e^{-sa} \qquad (3\ points).$$

For $u(a,0) = \delta(a-A)$ and $v(a,0) = 0$, Laplace transforming the PDEs gives

$$\bar{u}_a + (s+\mu)\bar{u} = \bar{v} + \delta(a-A) \qquad \& \qquad \bar{v}_a + (s+\mu+1)\bar{v} = 0,$$

Hence

$$\bar{v}(a,s) = \bar{v}(0,s)e^{-(s+\mu+1)a} \qquad \& \qquad [\bar{u}e^{(s+\mu)a}]_a = \bar{v}(0,s)e^{-a} + \delta(a-A)e^{(s+\mu)a},$$

or

$$\bar{u}(a,s) = \bar{v}(0,s)(1-e^{-a})e^{-(s+\mu)a} + H(a-A)e^{(s+\mu)(A-a)}.$$

Taking the transform of the birth condition:

$$\bar{v}(0,s) = \frac{3}{4}\int_0^\infty \bar{u}(a,s)\,da = \frac{3\bar{v}(0,s)}{4(s+\mu)(s+\mu+1)} + \frac{3}{4(s+\mu)},$$

or

$$\bar{v}(0,s) = \frac{3(s+\mu+1)}{4[(s+\mu+\frac{1}{2})^2-1]} = \frac{9}{16(s+\mu-\frac{1}{2})} + \frac{3}{16(s+\mu+\frac{3}{2})}$$

Thus, inverting the transform using the shifting theorem,

$$v(a,t) = \frac{1}{16}\left[9e^{\frac{1}{2}(t-a)} + 3e^{-\frac{3}{2}(t-a)}\right]e^{-\mu(t-a)-(\mu+1)a}H(t-a)$$

and

$$u(a,t) = (e^a-1)v(a,t) + e^{-\mu t}\delta(t-a+A)$$

(8 points.)

4. The characteristics equations and solution:

$$\frac{dx}{dt} = u^2 \quad \& \quad \frac{du}{dt} = 0 \qquad \longrightarrow \qquad x = x_0 + u^2 t \quad \& \quad u = f(x-u^2t).$$

Given the initial condition, there is a jump at $x=-1$ that broadens into an expansion fan with $x_0 = -1 = x - u^2t$ for $-1 < x < t-1$. Imposing the remainder of the initial condition then implies

$$u = \begin{cases} 0, & x < -1, \\ \sqrt{(1+x)/t}, & -1 \le x < t-1, \\ 1, & t-1 \le x < t, \\ \sqrt{(1-x)/(1-t)}, & t \le x < 1, \\ 0, & 1 \le x, \end{cases}$$

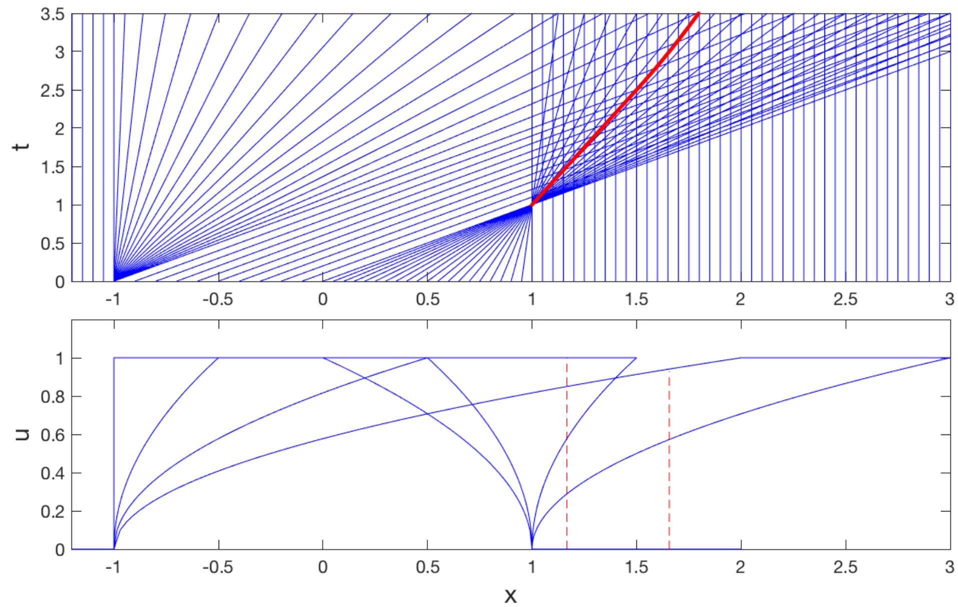
which breaks down for $t=1$ at $x=1$ when the third region shrinks to xero width and the solution there becomes vertical. *(4 points).*

Given that the flux is $J = \frac{1}{3}u^3$, the position of the shock $X(t)$ satisfies

$$\frac{dX}{dt} = \frac{1}{3}\frac{(u^+)^3-(u^-)^3}{u^+-u^-} = \frac{1}{3}[(u^+)^2 + u^+u^- + (u^-)^2].$$

Just after $t=1$, the shock jumps from $u^-=1$ to $u^+=0$, and so

$$X(t) = \frac{1}{3}(t-1) + 1 = \frac{1}{3}t + \frac{2}{3}.$$



But the right-hand edge of the fan collides with the shock when $x = t - 1 = X(t)$, or $t = \frac{5}{2}$, at $x = \frac{3}{2}$. Thereafter, we have $u^{-} \equiv \sqrt{(X+1)/t}$ and

$$\frac{dX}{dt} = \frac{(1+X)}{3t}, \qquad \text{or} \qquad X = \left(\frac{25t}{4}\right)^{\frac{1}{3}} - 1 \qquad\qquad (4 \textit{ points}).$$

Sketches: 3 points.