MATH 400: Applied PDEs. Final exam

Closed book exam; no calculators. Answer as much as you can; credit awarded for the best three answers. Each question is worth 14 points. Adequately explain the steps you take. *e.g.* if you use an expansion formula, say in one sentence why this is possible; if you quote a special function solution to an ODE, say why this is the correct one. Be as explicit as possible in giving your solutions.

1. (a) If $P_n^m(x)$ denotes an associated Legendre function (with integers n and m), show that

$$P_m^m(\cos\theta) = a_m \sin^m\theta,$$

where a_m is a constant that you should calculate.
(b) Solve Laplace's equation in spherical polar coordinates,

$$\nabla^2 u = \frac{1}{\rho^2}(\rho^2 u_\rho)_\rho + \frac{1}{\rho^2\sin\theta}(\sin\theta\, u_\theta)_\theta + \frac{1}{\rho^2\sin^2\theta}u_{\varphi\varphi} = 0,$$

over the shell, $1 < \rho < 2$, subject to

$$u \text{ regular for } \theta\to 0,\ \theta\to\pi, \quad u \text{ is } 2\pi-\text{periodic in } \varphi, \quad u(1,\theta,\varphi)=0, \quad u(2,\theta,\varphi) = f(\theta)\cos m\varphi,$$

writing your solution as an infinite series in which the coefficients are given by integrals of $f(\theta)$.
(c) Provide an explicit solution without any series or integrals for the special case with $f(\theta) = \sin^m\theta$.

2. (a) Given that $\mathcal{F}\{\mathcal{F}^{-1}\{\hat f(k)\}\} = \hat f(k)$, show that

$$\int_{-\infty}^{\infty} e^{-i(k-k')x}dx = 2\pi\delta(k-k')$$

and therefore establish that

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat f(k)\hat g(-k)dk$$

where $\mathcal{F}\{f(x)\} = \hat f(k)$ and $\mathcal{F}\{g(x)\} = \hat g(k)$. Show that

$$\frac{1}{\sqrt{\epsilon}}\mathcal{F}^{-1}\{e^{-|k|/\epsilon}\} = \frac{\sqrt{\epsilon}}{\pi(\epsilon^2x^2+1)},$$

where ϵ is a positive parameter.
(b) The intensity of light in an infinitely long fluorescent tube, $u(r,x)$ satisfies

$$u_{xx} + \frac{1}{r}(ru_r)_r = f(x), \quad -\infty < x < \infty, \quad 0 < r < 1,$$

$$u \text{ regular for } r\to 0, \quad u(1,x) = 0.$$

Find $\hat u(r,k)$, the Fourier transform in x of $u(r,x)$, in terms of $\hat f(k)$.
(c) The total power output of the tube is given by the integral

$$I = \int_{-\infty}^{\infty}[u_r(1,x)]^2\,dx.$$

Using the results from (a), express this power integral I as a single integral over k, first for general $f(x)$, and then in the special case, $f(x) = \sqrt{\epsilon}/[\pi(\epsilon^2x^2+1)]$. Evaluate this last integral for $\epsilon\to 0$.

3. (a) From the definition of the Laplace transform, establish the relations,

$$\mathcal{L}\{1\} = s^{-1}, \qquad \mathcal{L}\{f(t-a)H(t-a)\} = e^{-sa}\overline{f}(s) \qquad \text{and} \qquad \mathcal{L}\{\delta(t-a)\} = e^{-sa}$$

where $\overline{f}(s) = \mathcal{L}\{f(t)\}$, $a > 0$, $H(x)$ is the step function and $\delta(x)$ is the Dirac delta function.

(b) An age-structured model of the population at a summer holiday resort is based on the PDE,

$$u_t + u_a = ae^{-a} - \mu u, \qquad 0 \le a, t < \infty,$$

where $u(a,t)$ dictates the number of individuals with age a at time t, the term ae^{-a} denotes new tourist arrivals and μ is the (constant) rate of their departures. At the beginning of the season, there are only locals, $u(a,0) = U(a)$, and during the summer there are no births, $u(0,t) = 0$. Using the Laplace transform, find $u(a,t)$. Given that $U(a)$ is bounded, take the limit $t \to \infty$ to determine the steady state $u_\infty(a)$ that $u(a,t)$ approaches over long times. As a sanity check, confirm your result by solving the PDE for $u_\infty(a)$ directly.

4. (a) A model for traffic flow is based on the conservation law

$$\frac{d}{dt}\int_a^b u(x,t)\,dx + \big[u(1-u^2)\big]_{x=a}^{x=b} = 0,$$

where $[a,b]$ denotes an arbitrary interval. Turn this model into a PDE and then solve it using the method of characteristics for the initial condition $u(x,0) = f(x)$.

(b) Derive a formula for the speed of a shock in this model.

(c) If $f(x)$ contains a discontinuity at $x = 1$, with $u \to \frac{1}{2}$ to the left and $u \to 0$ to the right, argue that an expansion fan must appear for $t > 0$. Provide a solution for $u(x,t)$ within this fan.

(d) A lane of cars enters an empty highway, such that

$$u(x,0) = f(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & 0 \le x < 1, \\ 0, & x > 1. \end{cases}$$

Draw a characteristics diagram for this problem and demonstrate that, in addition to the expansion fan around $x = 1$, a shock must appear at $x = 0$ for $t > 0$. Find the position of the shock at later times, and add this position on your characteristics diagram. Sketch the solution for $t = 1$ and $t = \frac{5}{2}$.

Helpful information:

The Sturm-Liouville differential equation and expansion formulae:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda\sigma(x)y = 0, \qquad a \le x \le b,$$

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad c_n = \frac{\int_a^b f(x)y_n(x)\sigma(x)dx}{\int_a^b [y_n(x)]^2\sigma(x)dx}.$$

The ODE,

$$z^2y'' + zy' - (z^2 + n^2)y = 0,$$

with parameter $n = 0, 1, 2, ...$, has the regular solution $y(z) = AI_n(z)$ for $z \to 0$, where A is an arbitrary constant and $I_n(z)$ is a *modified Bessel function*. $I_0(z)$ is an even function with $I_0(0) = 1$ and $I_0''(0) = \frac{1}{2}$.

The associated Legendre differential equation, with regular solution $n = 1, 2, ...$ and $y = P_n^m(x)$ at $x = \pm 1$, is

$$(1 - x^2)y'' - 2xy' + n(n + 1)y - \frac{m^2 y}{1 - x^2} = 0; \qquad P_n^m(x) = (-1)^m(1 - x^2)^{m/2}\frac{d^m}{dx^m}[P_n(x)],$$

where $P_n(x)$ is a Legendre polynomial, which can be computed from Rodrigues' formula,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n; \qquad \int_{-1}^{1}[P_n^m(x)]^2 dx = \frac{2(n + m)!}{(2n + 1)(n - m)!}.$$

Fourier Transform:

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad \& \qquad f(x) = \mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}dk$$

Laplace Transform:

$$\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt \qquad \& \qquad f(t) = \mathcal{L}^{-1}\{\overline{f}(s)\} = \frac{1}{2\pi i}\int_C \overline{f}(s)e^{st}ds,$$

where C is the Bromwich contour

Cauchy's theorem: if $F(z)$ has a simple pole at $z = z_*$, but is otherwise analytic inside a closed contour $\mathcal{C}$,

$$\int_{\mathcal{C}} F(z)dz = 2\pi i\left[(z - z_*)F(z)\right]_{z \to z_*}.$$

Convolution:

$$f \circ g = \int_{-\infty}^{\infty} f(x')g(x - x')dx'.$$

Helpful trigonometric relations:

$$\cos(A + B) = \cos A\cos B - \sin A\sin B, \qquad \sin(A + B) = \sin A\cos B + \cos A\sin B.$$

MATH 400 – Solution

1. (14 points) (a) By combining Rodrigues formula with the expression relating $P_n^m(x)$ to $P_n(x)$, we establish the required result, with

$$a_m = (-1)^m \frac{(2m)!}{2^m m!} \qquad \textit{(2 points)}.$$

(b) To solve the PDE, we separate variables: $u = R(\rho)\Theta(\theta)\Phi(\varphi)$. The PDE can then be rearranged into

$$\sin^2\theta \left[\frac{(\rho^2 R')'}{R} + \frac{(\Theta' \sin\theta)'}{\Theta \sin\theta}\right] = -\frac{\Phi''}{\Phi}$$

which must therefore equal a separation constant, m^2, giving the ODE, $\Phi'' + m^2\Phi = 0$, with $m = 0, 1, ...$, and the usual functions ($\sin m\varphi$, $\cos m\varphi$ or a constant with $m = 0$) present in a Fourier series *(2 points)*. More re-arrangements now give

$$\frac{(\Theta' \sin\theta)'}{\Theta \sin\theta} - \frac{m^2}{\sin^2\theta} = -\frac{(\rho^2 R')'}{R} = -\lambda,$$

and another separation constant. With the quick change of variable $x = \cos\theta$, we find that $\Theta(x)$ satisfies the associated Legendre differential equation. Demanding that $\Theta(x)$ be regular at $\theta = 0$ and π, or $x = \pm 1$, gives $\lambda = n(n+1)$ for $n = 0, 1, 2, ...$ and $\Theta \propto P_n^m(x)$ *(2 points)*.

Last, the ODE for $R(\rho)$ is an Euler equation, with solutions ρ^α and

$$\alpha(\alpha+1) - n(n+1) = 0 \quad \to \quad \alpha = -n-1 \text{ or } \alpha = n.$$

Since we want $R(\rho)$ to vanish at $\rho = 1$, we set

$$R(\rho) = \textit{constant} \times (\rho^n - \rho^{-n-1}) \qquad \textit{(2 points)}.$$

The boundary condition at $\rho = 2$ tell us that we need only a single term of the full general solution; that with $\Phi \propto \cos m\varphi$. Hence,

$$u = \cos m\varphi \sum_{n=m}^{\infty} c_n(\rho^n - \rho^{-n-1}) P_n^m(x),$$

since $P_n^m = 0$ for $n < m$. Applying the boundary condition at $\rho = 2$ and using the SL expansion formula:

$$f(\theta) = \sum_{n=m}^{\infty} c_n(2^n - 2^{-n-1}) P_n^m(x) \qquad \Longrightarrow \qquad c_n = \frac{(2n+1)(n-m)!}{2(2^n - 2^{-n-1})(n+m)!} \int_{-1}^{1} f(\cos\theta) P_n^m(x) dx,$$

given that the associated Legendre functions satsify a Sturm-Liouville problem with weight function $\sigma(x) = 1$, and in view of the integral provided in the helpful information *(4 points)*.

(c) In the case $f(\theta) = \sin^m x$, the boundary condition is equivalent to $u(2, \theta, \varphi) = a_m^{-1} P_m^m(x) \cos m\varphi$. Hence

$$u = \frac{(\rho^n - \rho^{-n-1})}{a_m(2^n - 2^{-n-1})} P_n^m(x) \cos m\varphi. \qquad \textit{(2 points)}.$$

2. (14 points) (a) From the definition of the Fourier transform and its inverse,

$$\mathcal{F}\{\mathcal{F}^{-1}\{\hat{f}(k)\}\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(k-k')x} \hat{f}(k') \frac{dk'}{2\pi} dx$$

But this must equal $\hat{f}(k)$, establishing the first result *(1 point)*. Now consider the next integral. Inserting the definitions of the inverse transforms of $\hat{f}(k)$ and $\hat{g}(k)$ gives

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k+k')x}\hat{f}(k)\hat{g}(k')dkdk'dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(k+k')\hat{f}(k)\hat{g}(k')dkdk'$$

giving the second result *(2 points)*. Third,

$$\mathcal{F}^{-1}\{e^{-b|k|}\} = \int_{-\infty}^{\infty} e^{ikx-b|k|}\frac{dk}{2\pi} = \int_{-\infty}^{0} e^{ikx+bk}\frac{dk}{2\pi} + \int_{0}^{\infty} e^{ikx-bk}\frac{dk}{2\pi} = \frac{b}{\pi(b^2+x^2)}$$

giving the last result with $b = \epsilon^{-1}$. *(1 point)*

(b) Fourier transforming the PDE gives

$$\frac{1}{r}(r\hat{u}_r)_r - k^2\hat{u} = \hat{f}, \qquad \hat{u}(1,k) = 0, \quad \& \quad \hat{u}(r,k) \text{ regular for } r \to 0. \quad \textit{(2 points)}$$

But the ODE here is an inhomogeneous version of the modified Bessel equation with $n = 0$. The (regular) homogeneous solution is proportional to $I_0(kr)$, whereas any function of k alone can be used to furnish the particular solution $-\hat{f}/k^2$. Hence

$$\hat{u}(r,k) = \frac{\hat{f}}{k^2}\left[\frac{I_0(kr)}{I_0(k)} - 1\right]. \quad \textit{(4 points)}$$

(c) In view of the results established earlier, and because I_0 is even, we have

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\hat{u}_r(1,k,1)]^2 dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\hat{f}(k)\hat{f}(-k)}{k^2}\left[\frac{I_0'(k)}{I_0(k)}\right]^2 dk. \quad \textit{(2 points)}$$

For the Lorentzian, we have

$$I = \frac{1}{2\pi\epsilon}\int_{-\infty}^{\infty}\frac{e^{-2\epsilon^{-1}|k|}}{k^2}\left[\frac{I_0'(k)}{I_0(k)}\right]^2 dk \approx \frac{1}{8\pi}$$

when $\epsilon \to 0$, given $I'(k)/I(k) \approx \frac{1}{2}k$ for $k \ll 1$ *(2 points)*.

3. (14 points) (a) From the definitions (and as long as $\mathrm{Re}(s) > 0$)

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st}dt = \frac{1}{s},$$

$$\mathcal{L}\{f(t-a)H(t-a)\} = \int_a^{\infty} f(t-a)e^{-st}dt = \int_0^{\infty} f(\tau)e^{-s\tau-sa}dt = e^{-sa}\overline{f}(s)$$

and

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty}\delta(t-a)e^{-st}dt = e^{-sa} \qquad \textit{(3 points).}$$

(b) Laplace transforming the PDE gives

$$\bar{u}_a + (s+\mu)\bar{u} = U(a) + \frac{ae^{-a}}{s} \qquad \longrightarrow \qquad \bar{u}(a,s) = \int_0^a e^{(z-a)(s-\mu)}U(z)dz + \int_0^a \frac{z}{s}e^{(z-a)(s-\mu)-z}dz$$

(4 points). We invert the transform, using the results from (a) to find

$$u(a,t) = \int_0^a \delta(t-a+z)e^{\mu(a-z)}U(z)dz + \int_0^a H(t-a+z)e^{\mu(a-z)-z}zdz$$

$$= e^{-\mu t} U(a-t)H(a-t) + \frac{a(\mu-1)-1}{(\mu-1)^2} e^{-a} - \frac{z_*(\mu-1)-1}{(\mu-1)^2} e^{-\mu a + (\mu-1) z_*}$$

where $z_* = (a-t)H(a-t)$ (given that the delta-function and step of $H(t-a+z)$ only lie inside the integration range if $a > t$) *(4 points)*.

For $t \to \infty$, the first term disappears since $U(a)$ is bounded and $a < t$, whereas $z_* \to 0$, leaving

$$u \to \frac{a(\mu-1)-1}{(\mu-1)^2} e^{-a} + \frac{1}{(\mu-1)^2} e^{-\mu a} \qquad \text{(1 point)}.$$

Last we resolve the PDE for $u_\infty(a)$:

$$\frac{du_\infty}{da} + \mu u_\infty = a e^{a}, \quad u_\infty(0) = 0,$$

which gives the same result for $u_\infty(a)$ *(2 points)*.

4. (14 points) (a) We massage the integral form of the conservation law:

$$\int_a^b \left[u_t + (u - u^3)_x \right] \, dx = 0.$$

But the interval $[a, b]$ is arbitrary, and so the integrand itself must vanish, leading to the PDE

$$u_t + (1 - 3u^2) u_x = 0.$$

(1 point). The characteristics equations and solution:

$$\frac{dx}{dt} = 1 - 3u^2 \quad \& \quad \frac{du}{dt} = 0 \qquad \longrightarrow \qquad x = x_0 + (1-3u^2)t \quad \& \quad u = f(x_0) = f(x - t + 3u^2 t).$$

(2 points).

(b) We put $a < x_*(t) < b$, where $x = x_*(t)$ denotes the shock position. Then using the conservation law and Leibnitz's rule, we observe that

$$(u^- - u^+) \frac{dx_*}{dt} \int_a^{x_*} u_t \, dx + \int_{x_*}^b u_t \, dx = -[u - u^3]_a^b,$$

where $u^\pm$ are the limits of $u(x,t)$ from the right and left (respectively). If we now take the limit $a \to x_*^-$ and $b \to x_*^+$, we arrive at

$$\frac{dx_*}{dt} = 1 - [(u^-)^2 + u^- u^+ + (u^+)^2].$$

(2 points).

(c) The characteristics to the left of the jump at $x = 1$ have $x = x_0 + \frac{t}{4}$; those on the right have $x = x_0 + t$. A triangular region therefore opens up for $t > 0$ over which there are no characteristics from $x_0 < 1$ or $x_0 > 1$. Instead, we have $x_0 = 1$ here, or

$$x = 1 + (1 - 3u^2)t, \quad \text{or} \quad u = \sqrt{\frac{1 + t - x}{3t}}.$$

This solution spans the region

$$1 + \frac{t}{4} < x < 1 + t.$$

(3 points).

(d) For the specific initial condition given, we arrive at the characteristics diagram shown in the figure. Initially, the shock speed $3/4$, since $u^- = 0$ and $u^+ = \frac{1}{2}$. Hence $x_* = 3t/4$. But this shock hits the fan for $3t/4 = 1 + t/4$; *i.e.* at the point $(x,t) = (\frac{3}{2}, 2)$. From then on, $u^+ = \sqrt{\frac{1+t-x_*}{3t}}$. The shock position is now set by solving

$$\frac{dx_*}{dt} = 1 - \frac{1+t-x_*}{3t}, \quad x_*(2) = \frac{3}{2},$$

which gives

$$x_*(t) = t + 1 - \frac{3}{2}\left(\frac{t}{2}\right)^{\frac{1}{3}}.$$

(3 points, 3 points for the sketches).