Math 256. Sample final exam

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE $xy' + y^2 = 4$ with $y(1) = 0$ has the solution,

(a) $2 - 2x^4$ (b) $(x^4 - 1)/(1 + x)$ (c) $2(x^4 - 1)/(1 + x^4)$

(d) $e^{x-1} - 1$ (e) *None of the above.*

2. The system

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 2 & -1 \end{pmatrix} \mathbf{y}$$

has the general solution,

(a) $\mathbf{u}_1 + \mathbf{u}_2 e^t + \mathbf{u}_3 e^{-t}$ (b) $\mathbf{u}_1 e^t + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{-2t}$ (c) $\mathbf{u}_1 + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{3t}$

(d) $\mathbf{u}_1 e^t + \mathbf{u}_2 e^{-4t} + \mathbf{u}_3$ (e) *None of the above,*

for three constant vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

3. The inverse Laplace transform of

$$\bar{y}(s) = -\frac{8}{s(s^2 - 4)}$$

(a) $y(t) = 2t + e^t - e^{-t}$, (b) $y(t) = 2t - \cos 2t$, (c) $y(t) = 2 - e^{2t} - e^{-2t}$,

(d) $y(t) = 2 + \cos 2t$, (e) *None of the above.*

4. Which of the following is a solution to the PDE $u_{tt} = u_{xx}$:

(a) $u = \sin(x - t)$ (b) $u = \cos x \ \sin 4t$ (c) $u = \sin x \ e^{-t}$

(d) $u = e^t \ \sin x$ (e) *None of the above.*

5. A cool question

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(12 points)* The charge in a photon collector satisfies

$$q'' + 2q' + 10q = 0, \qquad q(0) = 0, \qquad q'(0) = A,$$

where A is the energy of an incident photon. Find $q(t)$. The collector is coupled to a detector whose signal $d(t)$ satisfies

$$d' + d = q, \qquad d(0) = 0.$$

For the detector to register the photon, $d(\pi)$ must exceed a threshold of 1. What incident energy A will trigger the detector?

2. *(12 points)* Write the ODEs

$$x' = 3x + y + e^t, \quad y' = 3y + x,$$

as a 2×2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. Find the solution if $x(0) = y(0) = 0$.

3. *(12 points)* From the definition of the Laplace transform and the properties of the delta function (plus an integration by parts) show that

$$\mathcal{L}\left\{t\delta'(t-c)\right\} = (cs-1)e^{-cs}, \qquad c > 0,$$

where $\delta'(t) = \frac{d\delta}{dt}$. Hence solve the ODE

$$\ddot{y} + 2\dot{y} + 5y = t\delta'(t-3), \qquad y(0) = \dot{y}(0) = 0.$$

4. *(16 points)* (a) Find the Fourier series of the function

$$f(x) = -x \text{ for } 0 < x < \pi, \qquad f(x) = -f(-x) \text{ for } -\pi < x < 0, \qquad \& \qquad f(x) = f(x + 2\pi).$$

(b) Find the $U(x)$ that satisfies $U_{xx} = 0$ with $U(0) = 0$ and $U(\pi) = \pi$.
(c) Using the method of separation of variables, solve

$$u_t = u_{xx}, \quad u(0,t) = 0, \quad u(\pi,t) = \pi, \quad u(x,0) = 0.$$

Math 256. Another one

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE $y' \tan x + y = 4$ with $y(\pi/2) = 2$ has the solution,

$$(a)\quad 4 - \frac{2}{\sin x} \qquad (b)\quad 2\sin x \qquad (c)\quad (x+2)\sin x + \cos x$$

$$(d)\quad \frac{4x}{\pi} - \cos x \qquad (e)\quad \textit{None of the above.}$$

2. The system

$$\mathbf{y}' = \begin{pmatrix} -3 & 0 & 1 \\ 0 & 1 & 0 \\ -5 & 0 & 3 \end{pmatrix} \mathbf{y}$$

has the general solution,

$$(a)\quad \mathbf{u}_1 + \mathbf{u}_2 e^{t} + \mathbf{u}_3 e^{-t} \qquad (b)\quad \mathbf{u}_1 e^{t} + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{-2t} \qquad (c)\quad \mathbf{u}_1 + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{3t}$$

$$(d)\quad \mathbf{u}_1 e^{t} + \mathbf{u}_2 e^{-4t} + \mathbf{u}_3 \qquad (e)\quad \textit{None of the above,}$$

for three constant vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

3. The inverse Laplace transform of

$$\bar{y}(s) = \frac{18}{s(s^2-9)}$$

is

$$(a)\quad y(t) = 2t + e^{3t} - e^{-3t}, \qquad (b)\quad y(t) = 2t - \cos 3t, \qquad (c)\quad y(t) = e^{3t} + e^{-3t} - 2,$$

$$(d)\quad y(t) = 2 + \cos 3t, \qquad (e)\quad \textit{None of the above.}$$

4. Which of the following is a solution to the PDE $u_{tt} = -u_{xxxx}$:

$$(a)\quad u = \sin(x-t) \qquad (b)\quad u = \cos x + \sin t \qquad (c)\quad u = \sin x\, e^{-t}$$

$$(d)\quad u = e^{t}\,\sin x \qquad (e)\quad \textit{None of the above.}$$

5. A cool question

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(12 points)* Solve

$$q'' + 4q' + 40q = 0, \qquad d' + 2d = q, \qquad f' = q(f + f^{-1}), \qquad q(0) = d(0) = f(0) = 0, \qquad q'(0) = 1.$$

2. *(12 points)* Write the ODEs

$$x' = x + 2y, \quad y' = 4x - y + 4e^{-t},$$

as a 2×2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. Find the solution if $x(0) = y(0) = 0$.

3. *(12 points)* Using the Laplace transform, solve the ODE

$$\ddot{y} + 4\dot{y} + 5y = (3t^3 - 2)\delta(t-1), \qquad y(0) = 0 \quad \dot{y}(0) = 1,$$

where $\delta(t)$ denotes Dirac's delta function.

4. *(16 points)* (a) Find the Fourier series of the function

$$f(x) = \pi \sin^2\left(\frac{x}{2}\right) - x \text{ for } 0 < x < \pi, \qquad f(x) = -f(-x) \text{ for } -\pi < x < 0, \qquad \& \qquad f(x) = f(x + 2\pi).$$

Hint: the helpful trig identities might prove handy.

(b) Find the $U(x)$ that satisfies $U_{xx} = 0$ with $U(0) = 0$ and $U(\pi) = \pi$.

(c) Using the method of separation of variables, solve

$$u_t = u_{xx}, \quad u(0,t) = 0, \quad u(\pi, t) = \pi, \quad u(x, 0) = \pi \sin^2\left(\frac{x}{2}\right).$$

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Helpful trig identities:

$$\sin 0 = \sin\pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),$$
$$\cos 0 = -\cos\pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$
$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$
$$\sin(2A) = 2\sin A\cos A, \quad \sin(A+B) = \sin A\cos B + \cos A\sin B,$$
$$\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,$$
$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

Useful Laplace Transforms:

$$f(t) \quad \to \quad \bar{f}(s)$$

$$1 \quad \to \quad 1/s$$

$$t^n, \quad n = 0, 1, 2, ... \quad \to \quad n!/s^{n+1}$$

$$e^{at} \quad \to \quad 1/(s-a)$$

$$\sin at \quad \to \quad a/(s^2+a^2)$$

$$\cos at \quad \to \quad s/(s^2+a^2)$$

$$t\sin at \quad \to \quad 2as/(s^2+a^2)^2$$

$$t\cos at \quad \to \quad (s^2-a^2)/(s^2+a^2)^2$$

$$y'(t) \quad \to \quad s\bar{y}(s) - y(0)$$

$$y''(t) \quad \to \quad s^2\bar{y}(s) - y'(0) - sy(0)$$

$$e^{at}f(t) \quad \to \quad \bar{f}(s-a)$$

$$f(t-a)H(t-a) \quad \to \quad e^{-as}\bar{f}(s)$$

$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$