Math 256. Final

Name:

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE $y' = xe^{x-y}$ with $y(0) = 0$ has the solution,

$$(a)\quad e^x \qquad (b)\quad \ln(2 - e^x) \qquad (c)\quad \ln[2 - (1 - x)e^x]$$

$$(d)\quad x \qquad (e)\quad \textit{None of the above.}$$

2. The system

$$\mathbf{y}'' = \begin{pmatrix} 1 & 6 & 7 \\ 0 & 4 & 0 \\ 0 & 3 & 2 \end{pmatrix} \mathbf{y}$$

has the general solution,

$$(a)\quad \mathbf{u}_1 e^{4t} + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{t} \qquad (b)\quad \mathbf{u}_1 e^{2t} + \mathbf{u}_2 e^{t} + \mathbf{u}_3 e^{-2t} \qquad (c)\quad \mathbf{u}_1 e^{-4t} + \mathbf{u}_2 e^{-2t} + \mathbf{u}_3 e^{-t}$$

$$(d)\quad \mathbf{u}_1 e^{-t} + \mathbf{u}_2 e^{t} + \mathbf{u}_3 e^{4t} \qquad (e)\quad \textit{None of the above,}$$

for three constant vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

3. The inverse Laplace transform of

$$\bar{y}(s) = \frac{4s}{(s^2 + 4s + 5)}$$

is

$$(a)\quad y(t) = 4e^{-t}\cos 2t - 2e^{-t}\sin 2t, \qquad (b)\quad y(t) = 4e^{-2t}\cos t, \qquad (c)\quad y(t) = 4e^{-t}\cos 2t,$$

$$(d)\quad y(t) = 4e^{-2t}(\cos t - 2\sin t), \qquad (e)\quad \textit{None of the above.}$$

4. Which of the following is a solution to the PDE $u_{xx} + 9u_{yy} - 8u = 0$:

$$(a)\quad u = \cos x\ \sin 3y \qquad (b)\quad u = \cos 3x\ \cos 3y \qquad (c)\quad u = \cos 3x\ e^{y}$$

$$(d)\quad u = \sin 3x\ \sin y \qquad (e)\quad u = \sin x\ e^{-y}.$$

5. Separation of variables

(*a*) *is a method for turning ODEs into PDEs*

(*b*) *uses the Laplace transform*

(*c*) *works for any PDE*

(*d*) *is cool if you like that sort of thing*

(*e*) *applies at resonance.*

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(12 points)* By popular demand, a mechanical oscillator satisfies the ODEs,

$$x' + 3x - y = 0, \quad y' - y + 4x = 0, \quad z' + z = y\cos t, \quad w' + wx = 0.$$

If $(x(0), y(0), z(0), w(0)) = (0, 1, 0, 0)$, what is the solution?

2. *(12 points)* Write the ODEs

$$x'' = 2y - x + 2\sin t, \quad y'' = 2x - 4y + \sin t,$$

as a 2×2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. What is special about the initial conditions $x(0) = y(0) = 0$, $x'(0) = -2$ and $y'(0) = -1$?

3. *(10 points)* From the definition of the Laplace transform, prove the two shifting theorems. Using Laplace transforms, solve the ODE

$$\ddot{y} + 2\dot{y} + y = e^{-t}[1 - H(t-1)],$$

with $y(0) = \dot{y}(0) = 1$, where $H(t)$ denotes the step function.

4. *(14 points)* Consider the PDE

$$u_t = u_{xx} + 1, \quad 0 < x < \pi, \qquad u_x(0,t) = \pi, \quad u_x(\pi,t) = 0, \quad u(x,0) = 0.$$

By integrating the PDE in x show that

$$\frac{d}{dt}\int_0^\pi u(x,t)dx = \int_0^\pi u_t dx = 0,$$

and so (for any t)

$$\int_0^\pi u(x,t)dx = 0,$$

in view of the initial condition. Use this information to fully specify the steady-state solution of the PDE, $u \to U(x)$. Hence solve the PDE.

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Helpful trig identities:

$$\begin{gathered}
\sin 0 = \sin\pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2), \\
\cos 0 = -\cos\pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0, \\
\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1, \\
\sin(2A) = 2\sin A\cos A, \quad \sin(A+B) = \sin A\cos B + \cos A\sin B, \\
\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B, \\
\sin(A+B) + \sin(A-B) = 2\sin A\cos B
\end{gathered}$$

Useful Laplace Transforms:

$$f(t) \quad \to \quad \bar{f}(s)$$

$$1 \quad \to \quad 1/s$$

$$t^n, \quad n = 0, 1, 2, ... \quad \to \quad n!/s^{n+1}$$

$$e^{at} \quad \to \quad 1/(s-a)$$

$$\sin at \quad \to \quad a/(s^2+a^2)$$

$$\cos at \quad \to \quad s/(s^2+a^2)$$

$$t\sin at \quad \to \quad 2as/(s^2+a^2)^2$$

$$t\cos at \quad \to \quad (s^2-a^2)/(s^2+a^2)^2$$

$$y'(t) \quad \to \quad s\bar{y}(s) - y(0)$$

$$y''(t) \quad \to \quad s^2\bar{y}(s) - y'(0) - sy(0)$$

$$e^{at}f(t) \quad \to \quad \bar{f}(s-a)$$

$$f(t-a)H(t-a) \quad \to \quad e^{-as}\bar{f}(s)$$

$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$