Math 256. Final

Name:

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE $y' = e^{x-y}$ with $y(0) = 0$ has the solution,

$(a)\quad e^x,\qquad (b)\quad \ln(2-e^x),\qquad (c)\quad \ln[2-(1-x)e^x],\qquad (d)\quad x,\qquad (e)\quad$ *None of the above.*

2. The general solution of the system,

$$\mathbf{y}'' = \begin{pmatrix} 2 & 6 & 7 \\ 0 & 4 & 0 \\ 0 & 3 & 1 \end{pmatrix} \mathbf{y} \ ,$$

is

$(a)\quad \mathbf{u}_1 e^{-4t} + \mathbf{u}_2 e^{-2t} + \mathbf{u}_3 e^{-t} \qquad (b)\quad \mathbf{u}_1 e^{4t} + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{t} \qquad (c)\quad \mathbf{u}_1 e^{2t} + \mathbf{u}_2 e^{t} + \mathbf{u}_3 e^{-2t}$

$(d)\quad \mathbf{u}_1 e^{-t} + \mathbf{u}_2 e^{t} + \mathbf{u}_3 e^{4t} \qquad (e)\quad$ *None of the above,*

for three constant vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

3. The inverse Laplace transform of the function,

$$\bar{y}(s) = \frac{4s}{(s^2+4s+5)} \ ,$$

is

$(a)\quad y(t) = 4e^{-t}\cos 2t - 2e^{-t}\sin 2t,\qquad (b)\quad y(t) = 4e^{-2t}\cos t,\qquad (c)\quad y(t) = 4e^{-t}\cos 2t,$

$(d)\quad y(t) = 4e^{-2t}(\cos t - 2\sin t),\qquad (e)\quad$ *None of the above.*

4. Which of the following is a solution to the PDE $u_{xx} + 9u_{yy} - 8u = 0$:

$(a)\quad u = \cos x\ \sin 3y \qquad (b)\quad u = \cos 3x\ \cos 3y \qquad (c)\quad u = \cos 3x\ e^y$

$(d)\quad u = \sin 3x\ \sin y \qquad (e)\quad u = \sin x\ e^{-y}.$

5. Separation of variables

(*a*) *is a method for turning ODEs into PDEs,* (*b*) *uses the Laplace transform,*

(*c*) *determines the integrating factor of an ODE,* (*d*) *is cool if you like that sort of thing,*

(*e*) *identifies resonant behaviour for a driven oscillator.*

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(12 points)* Professor X patents a machine that claims to consume power $x(t)$ whilst producing ice cream $z(t)$ and chocolate $w(t)$. He models the machine with the ODEs,

$$x'' - x = -2\sin t, \quad z'\cos t - xz = 1, \quad w' = xw^2.$$

For the initial conditions, $x(0) = 0$, $x'(0) = 1$, $z(0) = 0$ and $w(0) = 1$, what are the solutions? What happens to the ice cream and chocolate as time increases up to $\frac{1}{2}\pi$?

2. *(12 points)* Write the ODEs

$$x'' = 2y - 4x + \sin t, \quad y'' = 2x - y + 2\sin t,$$

as a 2×2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. What is special about the initial conditions $x(0) = y(0) = 0$, $x'(0) = -1$ and $y'(0) = -2$?

3. *(10 points)*
(a) From the definition of the Laplace transform, prove the two shifting theorems.
(b) The current in an electrical circuit satisfies the ODE,

$$\ddot{y} + 2\dot{y} + y = e^{-t}[1 - H(t-1)],$$

with $y(0) = A$ and $\dot{y}(0) = B$, where $H(t)$ denotes the step function and A and B are constants. Solve this ODE.
(c) For what values of A and B does the current vanish for $t > 1$?

4. *(14 points)*
(a) The function $f(x)$, defined on $0 \le x \le \pi$, is extended as an odd, $2\pi-$periodic function to $-\infty < x < \infty$. State the Fourier series of the extended function, giving the formulae for its coefficients as integrals over the original interval $[0, \pi]$.
(b) Consider the PDE

$$u_t = u_{xx} + \cos x, \quad 0 < x < \pi, \qquad u(0,t) = 1, \quad u(\pi,t) = -1, \quad u(x,0) = 0.$$

Find the steady-state solution, $u \to U(x)$. Then solve the PDE.

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

Helpful trig identities:

$$\sin 0 = \sin\pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),$$
$$\cos 0 = -\cos\pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$
$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$
$$\sin(2A) = 2\sin A\cos A, \quad \sin(A+B) = \sin A\cos B + \cos A\sin B,$$
$$\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,$$
$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

Useful Laplace Transforms:

$$f(t) \quad \rightarrow \quad \bar{f}(s)$$

$$1 \quad \rightarrow \quad 1/s$$

$$t^n, \quad n = 0, 1, 2, ... \quad \rightarrow \quad n!/s^{n+1}$$

$$e^{at} \quad \rightarrow \quad 1/(s-a)$$

$$\sin at \quad \rightarrow \quad a/(s^2+a^2)$$

$$\cos at \quad \rightarrow \quad s/(s^2+a^2)$$

$$t\sin at \quad \rightarrow \quad 2as/(s^2+a^2)^2$$

$$t\cos at \quad \rightarrow \quad (s^2-a^2)/(s^2+a^2)^2$$

$$y'(t) \quad \rightarrow \quad s\bar{y}(s) - y(0)$$

$$y''(t) \quad \rightarrow \quad s^2\bar{y}(s) - y'(0) - sy(0)$$

$$e^{at}f(t) \quad \rightarrow \quad \bar{f}(s-a)$$

$$f(t-a)H(t-a) \quad \rightarrow \quad e^{-as}\bar{f}(s)$$

$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$

Solutions:

Part I: (d), (e), (d), (e), (d)

1. The ODE is separable:

$$\int e^y dy = \int e^x dx + C \quad \longrightarrow \quad e^y = e^x + C \quad \text{or} \quad y = \ln[C + e^x]$$

where C must equal 0 from the starting condition. i.e. $y = x$. But by inspection (and with rather less effort) one can also see that this answer (d) clearly solves the ODE and starting condition.

2. The system is second-order and 3×3. Therefore, there must be six homogeneous solutions, two of which have the time dependence $\exp(\pm\sqrt{2}t)$ (one of the eigenvalues is 2).

3. We have

$$\frac{4s}{(s+2)^2+1} = \frac{4(s+2)}{(s+2)^2+1} - \frac{8}{(s+2)^2+1} = 4\mathcal{L}\{e^{-2t}\cos t\} - 8\mathcal{L}\{e^{-2t}\sin t\}$$

4. Answer (e) is the only one that successfully solves the PDE on substitution.

5. All the answers are silly, but (d) at least is not wrong.

Part II:

1. We divide and conquer, dealing with $x(t)$ first. This ODE has the homogeneous solutions, $A_1 e^t + A_2 e^{-t}$, and the particular solution $\sin t$. But the ICs demand that $A_1 = A_2 = 0$, so $x(t) = \sin t$. Inserting this into the $z-$equation gives

$$z' \cos t - z \sin t = 1 \quad \text{or} \quad \frac{d}{dt}(z \cos t) = 1 \quad \longrightarrow \quad z(t) = \frac{t + C}{\cos t} = \frac{t}{\cos t},$$

in view of the IC. Alternatively, but with more effort, one can divide by $\cos t$, find the integrating factor $I = \exp \int (-\sin t)/(\cos t) dt = \cos t$, and then solve the ODE (given $qI = 1$). Last,

$$w' + w^2 \sin t = 0,$$

which is separable:

$$\int \frac{dw}{w^2} = C + \int \sin t dt \quad \text{or} \quad -\frac{1}{w} = C - \cos t = -\cos t$$

given $w(0) = 1$. *i.e.* $w = \sec t$. The solutions for z and w diverge for $t \to \frac{1}{2}\pi$. Happy days.

2. The system is

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t$$

The eigenvalues of the matrix are 0 and -5, with eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, resectively. The particular solution is $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \sin t$ where

$$-d_1 = 2d_2 - 4d_1 + 1 \quad \& \quad -d_2 = 2d_1 - d_2 + 2,$$

which lead to $-\begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t$. The general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} (A + Bt) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} (C \cos \sqrt{5}t + D \sin \sqrt{5}t) - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t$$

The initial conditions give $A = B = C = D = 0$, implying no homogeneous solutions and a pure oscillation with unit frequency.

3. From the definition of the Laplace transform

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-(s-a)t} f(t)dt = \overline{f}(s)$$

$$\mathcal{L}\{H(t-a)f(t-a)\} = \int_a^\infty e^{-st} f(t-a)dt = e^{-as}\int_a^\infty e^{-s\tau} f(\tau)d\tau = e^{-as}\overline{f}(s)$$

The Laplace transform of the ODE gives

$$(s^2+2s+1)\overline{y}(s) - sA - B - 2A = \frac{1}{s+1} - \frac{e^{-s-1}}{s+1}$$

or

$$\overline{y}(s) = \frac{A}{s+1} + \frac{A+B}{(s+1)^2} + \frac{1}{(s+1)^3} - \frac{e^{-s-1}}{(s+1)^3}.$$

The inverse is

$$y(t) = Ae^{-t} + (A+B)te^{-t} + \frac{1}{2}t^2e^{-t} - \frac{1}{2}(t-1)^2e^{-t}H(t-1)$$

When $t > 1$, the RHS has the factor $A + (A+B)t - \frac{1}{2} + t$, which vanishes if $A = \frac{1}{2}$ and $B = -\frac{3}{2}$.

4.

$$(a) \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\, dx.$$

(b) The steady state solution, satisfying $U'' + \cos x = 0$, $U(0) = 1$ and $U(\pi) = -1$, is $U = \cos x$. Putting $v = u - U$ we then find the homogeneous problem,

$$v_t = v_{xx}, \quad v(0,t) = v(\pi,t) = 0, v(x,0) = -U(x).$$

Separating variables then gives

$$v = \sum_{n=1}^{\infty} b_n e^{-n^2t}\sin nx, \quad b_n = \frac{2}{\pi}\int_0^\pi (-\cos x)\sin nx\, dx,$$

which can be evaluated using a handy trig formula. Thence,

$$u(x,t) = \cos x - \sum_{n\ even}^{\infty} \frac{4ne^{-n^2t}\sin nx}{\pi(n^2-1)}.$$