Math 256: Differential Equation. Section 102 - Final Exam

Name and student number:

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE $y' \tan x = \tan y$ with $y(0) = 0$ has the solution,

$$(a) \quad \cos x, \qquad (b) \quad e^{\cos x}, \qquad (c) \quad x, \qquad (d) \quad \tan x, \qquad (e) \quad None\ of\ the\ above.$$

2. The general solution of the system,

$$\mathbf{y}' = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & 3 \\ 4 & 0 & 4 \end{pmatrix} \mathbf{y} \ ,$$

is

$$(a) \quad \mathbf{u}_1 e^{-4t} + \mathbf{u}_2 e^{-2t} + \mathbf{u}_3 e^{-t} \qquad (b) \quad \mathbf{u}_1 e^{4t} + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{t} \qquad (c) \quad \mathbf{u}_1 e^{2t} + \mathbf{u}_2 e^{t} + \mathbf{u}_3 e^{-2t}$$

$$(d) \quad \mathbf{u}_1 e^{-t} + \mathbf{u}_2 e^{t} + \mathbf{u}_3 e^{4t} \qquad (e) \quad None\ of\ the\ above,$$

for three constant vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

3. The inverse Laplace transform of the function,

$$\bar{y}(s) = \frac{4s}{(s^2 + 2s + 5)} \ ,$$

is

$$(a) \quad y(t) = 4e^{-t}\cos 2t - 2e^{-t}\sin 2t, \qquad (b) \quad y(t) = 4e^{-2t}\cos t, \qquad (c) \quad y(t) = 4e^{-t}\cos 2t,$$

$$(d) \quad y(t) = 4e^{-2t}(\cos t - 2\sin t), \qquad (e) \quad None\ of\ the\ above.$$

4. Which of the following is a solution to the PDE $u_{xx} + 9u_{yy} - 5u = 0$:

$$(a) \quad u = \cos x\ \sin 3y \qquad (b) \quad u = \cos 2x\ \cos 2y \qquad (c) \quad u = \cos x\ e^{2y}$$

$$(d) \quad u = \sin 3x\ \sin y \qquad (e) \quad u = \sin 2x\ e^{-y}.$$

5. The Laplace transform

(*a*) *is a method for turning ODEs into algebraic equations,*

(*b*) *is defined by an integral,*

(*c*) *is difficult to invert without a table of known transforms,*

(*d*) *is cool if you like that sort of thing,*

(*e*) *None of the above.*

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(12 points)* Solve the ODEs

$$x' - y = 2\cos t, \quad y' - x - y = \cos t, \quad yz' + xz + 1 = 0, \quad w' + 2xyw^2 = 0,$$

subject to the initial conditions, $x(0) = 0$, $y(0) = -1$, $z(0) = 0$ and $w(0) = 1$.

2. *(12 points)* Write the ODEs

$$x'' - 3y + 3x = 0, \quad y'' - x + y = 3\sin t,$$

as a 2×2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. For the initial conditions $x(0) = y(0) = 0$, $x'(0) = -3$ and $y'(0) = -2$, confirm that the solution oscillates with a single frequency that you should determine.

3. *(12 points)*
(a) From the definition of the Laplace transform, prove that

$$\mathcal{L}\{y''(t)\} = s^2\bar{y}(s) - y'(0) - sy(0), \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{and} \quad \mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\bar{f}(s).$$

(b) The current in an electrical circuit satisfies the ODE,

$$\ddot{y} + y = e^{-t}[1 - H(t-T)],$$

with $y(0) = A$ and $\dot{y}(0) = B$, where $H(t)$ denotes the step function and T, A and B are constants. Solve this ODE. Find the values of A and B for which the current vanishes for $t > T$. Show that this condition implies $A = -B = \frac{1}{2}$ if $T \gg 1$.

4. *(14 points)*
(a) The function $f(x) = \frac{1}{4}(1 - \cos 2x)$ is defined on $0 \le x \le \pi$, but is then extended as an odd, $2\pi-$periodic function. Sketch the extended function over the interval $-\pi < x < 3\pi$ and find its Fourier series.
(b) Solve the PDE

$$u_t = u_{xx} + \cos 2x, \quad 0 < x < \pi, \qquad u(0,t) = u(\pi,t) = u(x,0) = 0.$$

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Helpful trig identities:

$$\sin 0 = \sin\pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),$$
$$\cos 0 = -\cos\pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$
$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$
$$\sin(2A) = 2\sin A\cos A, \quad \sin(A+B) = \sin A\cos B + \cos A\sin B,$$
$$\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,$$
$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

Useful Laplace Transforms:

$$f(t) \quad \rightarrow \quad \bar{f}(s)$$

$$1 \quad \rightarrow \quad 1/s$$

$$t^n, \quad n = 0, 1, 2, ... \quad \rightarrow \quad n!/s^{n+1}$$

$$e^{at} \quad \rightarrow \quad 1/(s-a)$$

$$\sin at \quad \rightarrow \quad a/(s^2+a^2)$$

$$\cos at \quad \rightarrow \quad s/(s^2+a^2)$$

$$t\sin at \quad \rightarrow \quad 2as/(s^2+a^2)^2$$

$$t\cos at \quad \rightarrow \quad (s^2-a^2)/(s^2+a^2)^2$$

$$y'(t) \quad \rightarrow \quad s\bar{y}(s) - y(0)$$

$$y''(t) \quad \rightarrow \quad s^2\bar{y}(s) - y'(0) - sy(0)$$

$$e^{at}f(t) \quad \rightarrow \quad \bar{f}(s-a)$$

$$f(t-a)H(t-a) \quad \rightarrow \quad e^{-as}\bar{f}(s)$$

$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$

Solutions:

Part I: (c), (b), (a), (e), any of (a)-(d)

Part II:

1. We divide and conquer, dealing with $x(t)$ and $y(t)$ first. Eliminating $y = x' - 2\cos t$, we arrive at the ODE

$$x'' + 2\sin t - x - x' + 2\cos t = \cos t \quad \text{or} \quad x'' - x' - x = -2\sin t - \cos t$$

This ODE has the homogeneous solutions, $A_1 e^{m_1 t} + A_2 e^{m_2 t}$, and the particular solution $\sin t$, with

$$m_{1,2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

But the ICs demand that $A_1 + A_2 = 0$ and $-1 = x'(0) - 2$ or $x'(0) = m_1 A_1 + m_2 A_2 + 1 = 1$, demanding $A_1 = A_2 = 0$ and $x(t) = \sin t$ and $y(t) = -\cos t$. Inserting these into the $z-$equation gives

$$-z'\cos t + z\sin t = 1 \quad \text{or} \quad \frac{d}{dt}(z\cos t) = 1 \quad \longrightarrow \quad z(t) = \frac{t + C}{\cos t} = \frac{t}{\cos t},$$

in view of the IC. Alternatively, but with more effort, one can divide by $\cos t$, find the integrating factor $I = \exp\int(-\sin t)/(\cos t)dt = \cos t$, and then solve the ODE (given $qI = 1$). Last,

$$w' = 2w^2 \sin t\cos t = w^2 \sin 2t,$$

which is separable:

$$\int \frac{dw}{w^2} = C + \int \sin 2t dt \quad \text{or} \quad -\frac{1}{w} = C - \frac{1}{2}\cos 2t = -\frac{1}{2} - \frac{1}{2}\cos 2t$$

given $w(0) = 1$. *i.e.* $w = 2/(1 + \cos 2t)$.

2. The system is

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin t$$

The eigenvalues of the matrix are -4 and 0, with eigenvectors $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. The particular solution is $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \sin t$ where

$$-d_1 = 3d_2 - 3d_1 \quad \& \quad -d_2 = d_1 - d_2 + 3,$$

which lead to $-\begin{pmatrix} 3 \\ 2 \end{pmatrix} \sin t$. The general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(A + Bt) + \begin{pmatrix} 3 \\ -1 \end{pmatrix}(C\cos 2t + D\sin 2t) - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\sin t$$

The initial conditions give $A = B = C = D = 0$, implying no homogeneous solutions and a pure oscillation with unit frequency.

3. From the definition of the Laplace transform, as long as $\mathrm{Re}(s)$ is sufficiently positive and $y \to 0$ for $t \to \infty$,

$$\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st} y'(t)dt = [y(t)e^{-st}]_0^\infty + s\int_0^\infty e^{-st} y(t)dt = s\overline{y}(s) - y(0).$$

Replacing y' by y'' gives $\mathcal{L}\{y''\} = s\mathcal{L}\{y'\} - y'(0) = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$. Next,

$$\mathcal{L}\{e^{at}\} = \int_a^\infty e^{at-st}dt = -\left[\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}$$

Last,

$$\mathcal{L}\{H(t-a)f(t-a)\} = \int_a^\infty e^{-st}f(t-a)dt = e^{-as}\int_a^\infty e^{-s\tau}f(\tau)d\tau = e^{-as}\overline{f}(s)$$

The Laplace transform of the ODE gives

$$(s^2+1)\overline{y}(s) - sA - B = \frac{1}{s+1}(1-e^{-s-T})$$

or

$$\overline{y}(s) = \frac{As}{s^2+1} + \frac{B}{s^2+1} + \frac{1}{2}\left(\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right)(1-e^{-sT-T}).$$

The inverse is

$$y(t) = A\cos t + B\sin t + \frac{1}{2}(e^{-t} - \cos t + \sin t) - \frac{1}{2}[e^{T-t} - \cos(t-T) + \sin(t-T)]e^{-T}H(t-T)$$

When $t > T$, we have

$$y(t) = \frac{1}{2}\left[2A - 1 + e^{-T}(\cos T + \sin T)\right]\cos t + \frac{1}{2}\left[2B + 1 + e^{-T}(\sin T - \cos T)\right]\sin t$$

which vanishes if $2A = 1 - e^{-T}(\cos T + \sin T)$ and $2B = e^{-T}(\cos T - \sin T) - 1$. These imply $A = -B = \frac{1}{2}$ if $T \gg 1$.

4. (a) The extended function has the Fourier sine series,

$$f(x) = \sum_{n=1}^\infty b_n \sin nx,$$

with (using a handy trig formulae)

$$b_n = \frac{1}{2\pi}\int_0^\pi (1-\cos 2x)\sin nx\; dx = \frac{1}{\pi n} - \frac{n}{\pi(n^2-4)}, \quad n\ odd,$$

and $b_n = 0$ if n is even.

(b) The steady state solution, satisfying $U'' + \cos 2x = 0$, $U(0) = U(\pi) = 0$, is $U = \frac{1}{4}(\cos 2x - 1)$. Putting $v = u - U$ we then find the homogeneous problem,

$$v_t = v_{xx}, \quad v(0,t) = v(\pi,t) = 0, \quad v(x,0) = -U(x).$$

Separating variables then gives

$$v = \sum_{n=1}^\infty b_n e^{-n^2 t}\sin nx,$$

with b_n given in (a). Thence,

$$u(x,t) = \frac{1}{4}(1-\cos 2x) + \sum_{n=1}^\infty b_n e^{-n^2 t}\sin nx.$$