Math 256. Final

Name:

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE $3y' + y^{-2}e^x = y^{-2}$ with $y(0) = 0$ has the solution,

$(a)\quad e^x - 1 - x^3 \sin x \qquad (b)\quad e^x - e^{-x} - x^2 e^x \qquad (c)\quad (e^x - 1 - x^3)^3 + xe^{-2x}$

$(d)\quad -(e^x - 1 - x)^{1/3} \qquad (e)$ *None of the above.*

2. The system

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 6 & 7 \\ 0 & 3 & 2 \end{pmatrix} \mathbf{y}$$

has the general solution,

$(a)\quad \mathbf{u}_1 + \mathbf{u}_2 e^{6t} + \mathbf{u}_3 e^{t} \qquad (b)\quad \mathbf{u}_1 e^{7t} + \mathbf{u}_2 e^{t} + \mathbf{u}_3 e^{-2t} \qquad (c)\quad \mathbf{u}_1 + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{-t}$

$(d)\quad \mathbf{u}_1 e^{-t} + \mathbf{u}_2 + \mathbf{u}_3 e^{9t} \qquad (e)$ *None of the above,*

for three constant vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

3. The inverse Laplace transform of

$$\bar{y}(s) = \frac{4s}{(s^2 + 4s + 5)}$$

is

$(a)\quad y(t) = 4e^{-t}\cos 2t - 2e^{-t}\sin 2t, \qquad (b)\quad y(t) = 4e^{-2t}\cos t, \qquad (c)\quad y(t) = 4e^{-t}\cos 2t,$

$(d)\quad y(t) = 4e^{-2t}(\cos t - 2\sin t), \qquad (e)$ *None of the above.*

4. Which of the following is a solution to the PDE $u_{xx} + 9u_{yy} - 8u = 0$:

$(a)\quad u = \cos x\ \sin 3y \qquad (b)\quad u = \cos 3x\ \cos 3y \qquad (c)\quad u = \cos 3x\ e^y$

$(d)\quad u = \sin 3x\ \sin y \qquad (e)\quad u = \sin x\ e^{-y}.$

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(12 points)* A couple has attraction $x(t)$ but repulsion $y(t)$, leading to the degree of happiness $z(t)$, all satisfying the ODEs,

$$x' + x - y = 0, \quad y' + 3y + 5x = 0, \quad z' + 4z + xy = 0.$$

Determine what happens if the relationship starts with $(x(0), y(0), z(0)) = (0, 2, 0)$. Does happiness win out (*i.e.* $z(t)$ remain positive for $t \to \infty$)?

2. *(12 points)* Write the ODEs

$$x'' = 2x + y, \quad y'' = 5x - 2y + 8\sin t,$$

as a 2×2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. What is special about the initial conditions $x(0) = y(0) = x'(0) = 0$ and $y'(0) = -2$?

3. *(10 points)* From the definition of the Laplace transform, prove that $\mathcal{L}\{y''\} = s^2\bar{y}(s) - sy(0) - y'(0)$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\bar{f}(s)$. Find

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+2s+17)}\right\}$$

Using Laplace transforms, solve the ODE

$$\ddot{y} + 2\dot{y} + 17y = \delta(t-\pi)\cos t,$$

with $y(0) = 1$ and $\dot{y}(0) = -2$, where $\delta(t)$ is the delta-function.

4. *(14 points)* Solve

$$u_t = u_{xx}, \quad 0 < x < 3, \qquad u(0,t) = u(3,t) = 1, \quad u(x,0) = 0.$$

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

Helpful trig identities:

$$\sin 0 = \sin \pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),$$
$$\cos 0 = -\cos \pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$
$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$
$$\sin(2A) = 2\sin A\cos A, \quad \sin(A+B) = \sin A\cos B + \cos A\sin B,$$
$$\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,$$
$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

Useful Laplace Transforms:

$$f(t) \quad \rightarrow \quad \bar{f}(s)$$

$$1 \quad \rightarrow \quad 1/s$$

$$t^n, \quad n = 0, 1, 2, ... \quad \rightarrow \quad n!/s^{n+1}$$

$$e^{at} \quad \rightarrow \quad 1/(s-a)$$

$$\sin at \quad \rightarrow \quad a/(s^2+a^2)$$

$$\cos at \quad \rightarrow \quad s/(s^2+a^2)$$

$$t\sin at \quad \rightarrow \quad 2as/(s^2+a^2)^2$$

$$t\cos at \quad \rightarrow \quad (s^2-a^2)/(s^2+a^2)^2$$

$$y'(t) \quad \rightarrow \quad s\bar{y}(s) - y(0)$$

$$y''(t) \quad \rightarrow \quad s^2\bar{y}(s) - y'(0) - sy(0)$$

$$e^{at}f(t) \quad \rightarrow \quad \bar{f}(s-a)$$

$$f(t-a)H(t-a) \quad \rightarrow \quad e^{-as}\bar{f}(s)$$

$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$

Solutions:

Part I: (d), (d), (d), (e)

Part II:

1. $x = e^{-2t}\sin 2t$, $y = (2\cos 2t - \sin 2t)e^{-2t}$ and $z = \frac{1}{8}(4t - 2 + 2\cos 4t - \sin 4t)e^{-4t}$, so $z(t)$ remains positive for $t \to \infty$, but get's awfully small (a cynical view of marriage?)

2. The eigenvalues are ± 3 giving

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (Ae^{\sqrt{3}t} + Be^{-\sqrt{3}t}) + \begin{pmatrix} 1 \\ -5 \end{pmatrix} (C\cos\sqrt{3}t + D\sin\sqrt{3}t) + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \sin t$$

The initial conditions give $A = B = C = 0$ and $D = \sqrt{3}/3$; *i.e.* pure oscillations.

3. The inverse Laplace transform gives

$$y = e^{-t}\cos 4t - \frac{1}{4}e^{-t}\sin 4t - \frac{1}{4}e^{-(t-\pi)}\sin 4(t-\pi)H(t-\pi).$$

4. We put $u = v(x,t) - 1$ to homogenize the boundary conditions, then separate variables, giving

$$u(x,t) = 1 + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/9} \sin\left(\frac{n\pi x}{3}\right),$$

with

$$b_n = \frac{2}{n\pi}[(-1)^n - 1].$$

Math 256. Final variation

Part I

1. The ODE $4y' + e^x y^{-3} = y^{-3}$ with $y(0) = 0$ has the solution,

(a) $e^x - 1 - x^3 \sin x$ (b) $e^x - e^{-x} - x^2 e^x$ (c) $(e^x - 1 - x^3)^3 + xe^{-2x}$

(d) $(1 - e^x + x)^{1/4}$ (e) *None of the above.*

2. The system

$$\mathbf{y}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 4 \\ 3 & 1 & 0 \end{pmatrix} \mathbf{y}$$

has the general solution,

(a) $\mathbf{u}_1 + \mathbf{u}_2 e^{6t} + \mathbf{u}_3 e^t$ (b) $\mathbf{u}_1 e^{7t} + \mathbf{u}_2 e^t + \mathbf{u}_3 e^{-2t}$ (c) $\mathbf{u}_1 e^t + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{-2t}$

(d) $\mathbf{u}_1 e^{-t} + \mathbf{u}_2 + \mathbf{u}_3 e^{9t}$ (e) *None of the above,*

for three constant vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

3. The inverse Laplace transform of

$$\bar{y}(s) = \frac{4}{(s^2 + 4s + 5)}$$

is

(a) $y(t) = 4e^{-t} \sin 2t - 2e^{-t} \sin 2t$, (b) $y(t) = 4e^{-2t} \sin t$, (c) $y(t) = 4e^{-t} \sin 2t$,

(d) $y(t) = 4e^{-2t}(\cos t - 2\sin t)$, (e) *None of the above.*

4. Which of the following is a solution to the PDE $u_{xx} + 9u_{yy} - 8u_{xy} = 0$:

(a) $u = \cos x \ \sin 3y$ (b) $u = \cos 3x \ \cos 3y$ (c) $u = \cos 3x \ e^y$

(d) $u = \sin x \ e^{-y}$, (e) *None of the above.*

Part II

1. *(12 points)* Solve the ODEs,

$$x' + x - y = 0, \quad y' + 3y + 10x = 0, \quad z' + 4z + xy = 0.$$

with $(x(0), y(0), z(0)) = (0, 3, 0)$.

2. *(12 points)* Write the ODEs

$$x'' = 5y - 2x + 8\sin t, \quad y'' = 2y + x,$$

as a 2×2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. What is special about the initial conditions $x(0) = y(0) = y'(0) = 0$ and $x'(0) = -2$?

3. *(10 points)* From the definition of the Laplace transform, prove that $\mathcal{L}\{y''\} = s^2\bar{y}(s) - sy(0) - y'(0)$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\bar{f}(s)$. Find

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2 - 2s + 10)}\right\}$$

Using Laplace transforms, solve the ODE

$$\ddot{y} - 2\dot{y} + 10y = \delta(t - \pi)\cos 3t,$$

with $y(0) = \dot{y}(0) = 1$, where $\delta(t)$ is the delta-function.

4. *(14 points)* Solve

$$u_t = 9u_{xx}, \quad 0 < x < 4, \qquad u(0,t) = u(4,t) = 2, \quad u(x,0) = 1.$$

Math 256. Another Final variation

1. The ODE $y' - 4x + 3x^2y = 0$ with $y(0) = 0$ has the solution,

$(a)\quad \dfrac{2x^2-1}{x^3} \qquad (b)\quad \dfrac{1-2x^2}{x^4} \qquad (c)\quad \dfrac{5x^2-4}{3x^2} \qquad (d)\quad \dfrac{3x^2-2}{3x^3} \qquad (e)\quad$ *None of the above.*

2. For certain initial conditions, the system

$$\mathbf{y}'' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 4 \\ 3 & 4 & 0 \end{pmatrix} \mathbf{y}$$

has the solution,

$(a)\quad \mathbf{u}_1 + \mathbf{u}_2e^{4t} + \mathbf{u}_3e^{-4t} \qquad (b)\quad \mathbf{u}_1e^{4t} + \mathbf{u}_2e^{2t} + \mathbf{u}_3e^{-4t} \qquad (c)\quad \mathbf{u}_1e^{t} + \mathbf{u}_2e^{4t} + \mathbf{u}_3e^{-4t}$

$(d)\quad \mathbf{u}_1e^{-t} + \mathbf{u}_2e^{-2t} + \mathbf{u}_3e^{2t} \qquad (e)\quad$ *None of the above,*

for three constant vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

3. The inverse Laplace transform of

$$\bar{y}(s) = \frac{6-s}{(s^2+4)(s-1)}$$

is

$(a)\quad y(t) = e^{-t} + \cos 2t, \qquad (b)\quad y(t) = e^{-t} - \cos 2t, \qquad (c)\quad y(t) = e^{-t} - \sin 2t,$

$(d)\quad y(t) = e^{-t} - \cos 2t - \sin 2t, \qquad (e)\quad$ *None of the above.*

4. Which of the following is a solution to the PDE $u_t - u_{xx} + 3u = 0$:

$(a)\quad u = \cos t\ \sin x \qquad (b)\quad u = e^t\ \cos x \qquad (c)\quad u = e^{-4t}\ \sin x$

$(d)\quad u = \sin x + e^t, \qquad (e)\quad$ *None of the above.*

5. The Laplace transform of $te^t \sin t$ is

$(a)\quad \dfrac{s+2}{(s^2+1)^2} \qquad (b)\quad \dfrac{s+1}{(s^2+1)^2} \qquad (c)\quad \dfrac{s}{(s^2+1)^2+1} \qquad (d)\quad \dfrac{s}{[(s^2+1)^2+1]^2},$

$(e)\quad$ *None of the above.*

Part II

1. *(12 points)* A satellite satisfies the equations of motion,

$$x' - 2x + 2y = 0, \quad y' + 2y - 4x = 0, \quad z' = 4xz(y-x),$$

Find the path taken by the satellite if it starts at the point $(x(0), y(0), z(0)) = (1, 1, 1)$. Does the satellite eventually fall into the sun at $x = y = z = 0$?

2. *(12 points)* Write the ODEs

$$x'' - 2y + 2x = \cos t, \quad y'' + 2y - 2x = 0,$$

as a 2×2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system.

3. *(10 points)* From the definition of the Laplace transform and for constant a, find

$$\mathcal{L}\{y'\}, \quad \mathcal{L}\{y''\}, \quad \mathcal{L}\{y(t-a)H(t-a)\},$$

in terms of $\overline{y}(s) = \mathcal{L}\{y(t)\}$, $y(0)$ and $y'(0)$. Using Laplace transforms, solve the ODE

$$\ddot{y} - 4\dot{y} + 13y = (t^2 + t + 1)\delta(t-1),$$

with $y(0) = 1$ and $\dot{y}(0) = 4$, where $\delta(t)$ is the delta-function.

4. *(14 points)* Solve

$$9u_t = u_{xx}, \quad 0 < x < 4, \qquad u(0,t) = 0, \quad u(4,t) = 4, \quad u(x,0) = 0.$$