MATH 400: Applied PDEs. Final exam review

The final is closed book; no calculators.

- You can answer as much as you can; but credit will be awarded for the best three answers. *i.e.* 100% equals three complete answers.

- Each question is worth 14 points. (Questions with slightly different weight looked to be making people uncomfortable, and is also a tad inelegant.)

- Adequately explain the steps you take. *e.g.* if you use an expansion formula, say in one sentence why this is possible; if you quote a special function solution to an ODE, say why this is the correct one.

- Be as explicit as possible in giving your solutions. This is not strictly necessary for complete marks, but might help if the solution is used later.

- The helpful information on the next page will be provided.

Helpful information:

The Sturm-Liouville differential equation and expansion formulae:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda\sigma(x)y = 0, \qquad a \le x \le b,$$

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad c_n = \frac{\int_a^b f(x)y_n(x)\sigma(x)dx}{\int_a^b [y_n(x)]^2\sigma(x)dx}.$$

The ODE,

$$z^2y'' + zy' - (z^2 + \nu^2)y = 0,$$

with parameters ω and $n = 0, 1, 2, ...$, has the regular solution $y(z) = AI_n(z)$ for $z \to 0$, where A is an arbitrary constant and $I_n(z)$ is a *modified Bessel function*. $I_0(z)$ is an even function with $I_0(0) = 1$ and $I_0''(0) = \frac{1}{2}$.

The ODE

$$x^2y'' + (1 - 2\alpha)xy' - (\omega^2\beta^2x^{2\beta} - \alpha^2 + \nu^2\beta^2)y = 0,$$

with parameters α, $\omega > 0$, β and $\nu > 0$, has the solutions $x^\alpha I_\nu(\omega x^\beta)$ and $x^\alpha K_\nu(\omega x^\beta)$, where $K_\nu(z)$ is the other *modified Bessel function*. For $z \gg 1$, $I_\nu(z) \approx e^z/\sqrt{2\pi z}$ and $K_\nu(z) \approx e^{-z}\sqrt{\pi/(2z)}$.

The associated Legendre differential equation, with regular solution $n = 1, 2, ...$ and $y = P_n^m(x)$ at $x = \pm 1$, is

$$(1 - x^2)y'' - 2xy' + n(n+1)y - \frac{m^2y}{1 - x^2} = 0; \qquad P_n^m(x) = (-1)^m(1 - x^2)^{m/2}\frac{d^m}{dx^m}[P_n(x)],$$

where $P_n(x)$ is a Legendre polynomial, which can be computed from Rodrigues' formula,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n; \qquad \int_{-1}^{1}[P_n^m(n)]^2dx = \frac{2(n+m)!}{(2n+1)(n-m)!}.$$

Fourier Transform:

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad \& \qquad f(x) = \mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}dk$$

Laplace Transform:

$$\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt \qquad \& \qquad f(t) = \mathcal{L}^{-1}\{\overline{f}(s)\} = \frac{1}{2\pi i}\int_C \overline{f}(s)e^{st}ds,$$

where C is the Bromwich contour

Cauchy's theorem: if $F(z)$ has a simple pole at $z = z_*$, but is otherwise analytic inside a closed contour $\mathcal{C}$,

$$\int_{\mathcal{C}} F(z)dz = 2\pi i\left[(z - z_*)F(z)\right]_{z \to z_*}.$$

Convolution:

$$f \circ g = \int_{-\infty}^{\infty} f(x')g(x - x')dx'.$$

Helpful trigonometric relations:

$$\cos(A + B) = \cos A\cos B - \sin A\sin B, \qquad \sin(A + B) = \sin A\cos B + \cos A\sin B.$$

1. Solve the PDE
$$\frac{1}{\rho^2}(\rho^2 u_\rho)_\rho + \frac{1}{\rho^2 \sin\theta}(\sin\theta\, u_\theta)_\theta - \frac{u}{\rho^2 \sin^2\theta} = 0,$$
for $\rho > 1$, subject to
$$u \text{ regular for } \rho \to \infty,\ \theta \to 0,\ \theta \to \pi, \quad u(1,\theta,\varphi) = \sin^3\theta.$$

Solution: We separate variables: $u = R(\rho)\Theta(\theta)$. The PDE can then be re-arranged into
$$\frac{(\Theta' \sin\theta)'}{\Theta \sin\theta} - \frac{1}{\sin^2\theta} = -\frac{(\rho^2 R')'}{R},$$
which is a funk of θ on the left and a funk of ρ on the right, and so must therefore equal a separation constant that we denote by $-\lambda$. With the quick change of variable $x = \cos\theta$, we find that $\Theta(x)$ satisfies the associated Legendre differential equation with $m = 1$. Demanding that $\Theta(x)$ be regular at $\theta = 0$ and π, or $x = \pm 1$, gives $\lambda = n(n+1)$ for $n = 0, 1, 2, ...$ and $\Theta \propto P_n^m(x)$. Next, the ODE for $R(\rho)$ is an Euler equation, with solutions ρ^α and
$$\alpha(\alpha+1) - n(n+1) = 0 \quad \to \quad \alpha = -n-1$$
since we want $R(\rho)$ to be regular for $\rho \to \infty$.

Altogether, we now have a general solution
$$u = \sum_{n=0}^{\infty} c_n \rho^{-n-1} P_n^m(x), \quad c_n = \frac{(2n+1)}{2n(n+1)} \int_0^\pi f(\theta) P_n^1(\cos\theta) \sin\theta \, d\theta,$$
where the coefficient follows from imposing the boundary condition $u(1,\theta) = f(\theta)$ and observing that $\Theta(\theta)$ satisfies a Sturm-Liouville problem with weight function $\sigma(\theta) = \sin\theta$.

The helpful information tells us that
$$P_1^1 = -\sqrt{1-x^2}, \quad P_2^1 = -3x\sqrt{1-x^2}, \quad P_3^1 = \frac{3}{2}(1-5x^2)\sqrt{1-x^2}.$$
But
$$\sin^3\theta = (1-x^2)\sqrt{1-x^2} = \frac{2}{15}P_3^1 - \frac{4}{5}P_1^1.$$
Without performing any of the integrals for c_n, the solution is therefore
$$u = \frac{2}{15}\rho^{-4} P_3^1(x) - \frac{4}{15}\rho^{-2} P_1^1(x).$$

2. For the Fourier transform, establish the results,

$$\mathcal{F}^{-1}\{e^{-b|k|}\} = \frac{b}{\pi(x^2+b^2)} \qquad \& \qquad f \circ g = \mathcal{F}^{-1}\{\hat{f}\hat{g}\},$$

where $b > 0$, $\mathcal{F}\{f\} = \hat{f}(k)$, $\mathcal{F}\{g\} = \hat{g}(k)$ and $f \circ g$ is a convolution.

For parameters $a \geq 0$ and $\omega > 0$, write down the solution, $R(r)$, to

$$R'' + \frac{a}{r}R' - \omega^2 R = 0 \tag{1}$$

in terms of modified Bessel functions. Use your result and the Fourier transform in x to solve

$$u_{xx} + \frac{1}{r^2}(r^2 u_r)_r = 0,$$

$$1 \leq r < \infty, \quad -\infty < x < \infty, \quad u(x,1) = f(x), \quad u \to 0 \text{ for } r \to \infty \ \& \ |x| \to \infty,$$

expressing your answer in the form, $u(x,r) = \mathcal{F}^{-1}\{\hat{u}(k,r)\}$, where $\hat{u}(k,r)$ is a transform function that you should calculate.

If $a = 2$, verify that the solutions to (1) can be written more explicitly as $R \propto r^{-a/2}e^{\pm\omega r}$ (rather than modified Bessel functions). Hence reduce your solution to the PDE for $u(x,r)$ to a convolution integral.

Solution: From the definition of the Fourier transform and its inverse,

$$\mathcal{F}^{-1}\{e^{-b|k|}\} = \int_{-\infty}^{\infty} e^{ikx-b|k|}\frac{dk}{2\pi} = \int_{-\infty}^{0} e^{ikx+bk}\frac{dk}{2\pi} + \int_{0}^{\infty} e^{ikx-bk}\frac{dk}{2\pi} = \frac{1}{2\pi(b+ix)} + \frac{1}{2\pi(b-ix)}$$

and

$$\mathcal{F}\{f \circ g\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ikx}g(x-x')f(x')dxdx' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ikz-ikx'}g(z)f(x')dx'dz,$$

giving both results. The ODE is a special case of the general ODE quoted in the helpful information for $1-2\alpha = a$, $\alpha^2 = \nu^2\beta^2$ and $\beta = 1$. Hence the solutions are

$$R \propto r^{\nu}I_{\nu}(\omega r) \quad \text{or} \quad r^{\nu}K_{\nu}(\omega r), \quad \text{with } \nu = \tfrac{1}{2}|1-a|.$$

Fourier transforming the PDE and boundary conditions gives

$$\hat{u}_{rr} + \frac{2}{r}\hat{u}_r - k^2\hat{u} = 0, \qquad \hat{u}(k,1) = \hat{f}(k) \quad \& \quad \hat{u}(k,r) \to 0 \text{ for } r \to \infty,$$

Hence

$$\hat{u}(k,r) = r^{-1/2}\hat{f}(k)\frac{K_{1/2}(r|k|)}{K_{1/2}(|k|)},$$

given the large-argument form of the modified Bessel functions. Consequently,

$$u(x,r) = r^{-1/2}\mathcal{F}^{-1}\left\{\hat{f}(k)\frac{K_{1/2}(r|k|)}{K_{1/2}(|k|)}\right\}$$

For $a = 2$, we may substitute $e^{\pm\omega r}/r$ into the ODE to verify that these are indeed the solutions. We therefore have instead (after using the results established earlier)

$$u(x,r) = r^{-1}\mathcal{F}^{-1}\left\{\hat{f}(k)e^{-(r-1)|k|}\right\} \equiv \frac{r-1}{\pi r}\int_{-\infty}^{\infty}\frac{f(x')dx'}{(r-1)^2+(x-x')^2}$$

3. Establish the relations,

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad \mathcal{L}\{f'(t)\} = s\overline{f}(s) - f(0) \qquad \& \qquad \mathcal{L}\{f(t-a)H(t-a)\} = e^{-sa}\overline{f}(s),$$

where $\overline{f}(s) = \mathcal{L}\{f(t)\}$, $a > 0$ and $H(x)$ is the Heaviside step function.

An age-structured model of a population is based on the PDE,

$$u_t + u_a = -\mu(a)u, \qquad 0 \le a, t < \infty, \qquad u(a,0) = S(a) = \exp\left[-\int_0^a \mu(a')da'\right], \qquad u(0,t) = U(t),$$

where $u(a,t)$ denotes the number of individuals with age a at time t. Solve this problem using the Laplace transform.

Establish that the population reaches a steady state for large times if the number of individuals born with age $a = 0$ is constant: $U(t) = U_\infty = \textit{constant}$. If, on the other hand, the spontaneous fragmentation of individuals with age $a = A$ creates N new individuals of zero age (so that $u(0,t) = Nu(A,t)$), find a relation between $U(t)$ and $U(t-A)$ that must be satisfied by the solution of the PDE. Use this relation to argue that

$$U(t + nA) = \lambda^n U(t), \quad n = 1, 2, ...,$$

where λ is a parameter that you should determine. Hence provide conditions for which the population can achieve a steady state, dies out, or explodes.

Solution: From the definitions (and as long as $\mathrm{Re}(s) > 0$ and $f(t)$ is bounded for $t \to \infty$),

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \frac{1}{s}, \qquad \mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt = -f(0) + s\int_0^\infty f(t)e^{-st}dt = s\overline{f}(s) - f(0)$$

and

$$\mathcal{L}\{f(t-a)H(t-a)\} = \int_a^\infty f(t-a)e^{-st}dt = \int_0^\infty f(\tau)e^{-s\tau - sa}d\tau = e^{-sa}\overline{f}(s).$$

Laplace transforming the PDE gives

$$\bar{u}_a = S(a) - [s + \mu(a)]\bar{u} \quad \text{or} \quad (e^{sa}S^{-1}\bar{u})_a = e^{sa} \qquad \longrightarrow \qquad \bar{u}(a,s) = \frac{1 - e^{-sa}}{s}S(a) + \bar{U}(s)S(a)e^{-as}$$

Hence, using the shifting theorem established above,

$$u(a,t) = S(a)\left[1 - H(t-a) + U(t-a)H(t-a)\right]$$

When we take the limit $t \to \infty$, the step functions can be set to unity and we find the steady state solution, $u = U_\infty S(a)$, when $U(t) \to U_\infty$.

If $U(t) = Nu(A,t)$, however, then we may use the PDE solution to write

$$U(t) = NS(A)\left[1 - H(t-A) + U(t-A)H(t-A)\right].$$

For $t > A$, we again replace the step functions to arrive at

$$U(t) = NS(A)U(t-A), \qquad \text{or } U(t+A) = \lambda U(t) \qquad \text{with } \lambda = NS(A),$$

which establishes the required result on iteration. We observe that the solution becomes time-independent when $NS(A) = 1$, dies out if $NS(A) < 1$, and explodes for $NS(A) > 1$

4. Using the method of characteristics, solve the PDE

$$u_t + \left(u - \frac{1}{2}u^2\right)_x = 0, \qquad u(x,0) = f(x) = \begin{cases} 0, & x < -1, \\ 1, & -1 \le x < 0, \\ 0, & 0 \le x. \end{cases}$$

Draw a characteristics diagram and demonstrate that a shock and an expansion fan must appear. Provide sketches of the multi-valued solution predicted by the characteristics method at $t = 1$ and $t = 3$. On your sketches, indicate how the removal of the multi-valued sections by the insertion of a shock corresponds to an equal areas rule. How does that rule relate to the original PDE?

Solution: The characteristics equations and solution:

$$\frac{dx}{dt} = 1 - u \quad \& \quad \frac{du}{dt} = 0 \qquad \longrightarrow \qquad x = x_0 + (1-u)t \quad \& \quad u = f(x_0) = f(x - t + ut).$$

Given the form of the initial condition, we have $x = x_0 + t$ and $u = 0$ for $x_0 < -1$ or $x_0 > 0$, and $x = x_0$ and $u = 1$ over $-1 < x_0 < 0$. See the figure. The crossed characteristics above $(x,t) = (-1,0)$ imply that a shock appears instantly there. On the other hand, the gap above $(x,t) = (-1,0)$ must be filled with an expansion fan.

The PDE has a flux $J = u - \frac{1}{2}u^2$, and the position of any discontinuous solution (shock) evolves according to

$$\frac{dX}{dt} = \frac{J^+ - J^-}{u^+ - u^-} \equiv \frac{2u^+ - (u^+)^2 - 2u^- + (u^-)^2}{2(u^+ - u^-)} = 1 - \frac{1}{2}u^+,$$

where $\pm$ superscripts refer to the limits values to the shock position from the right and left, respectively, and given that $u^- = 0$. To begin, $u^+ = 1$, and so

$$X = -1 + \frac{t}{2}$$

But when $t = 2$, the shock hits the fan, and u^+ no longer equals 1, forcing the shock speed to change.

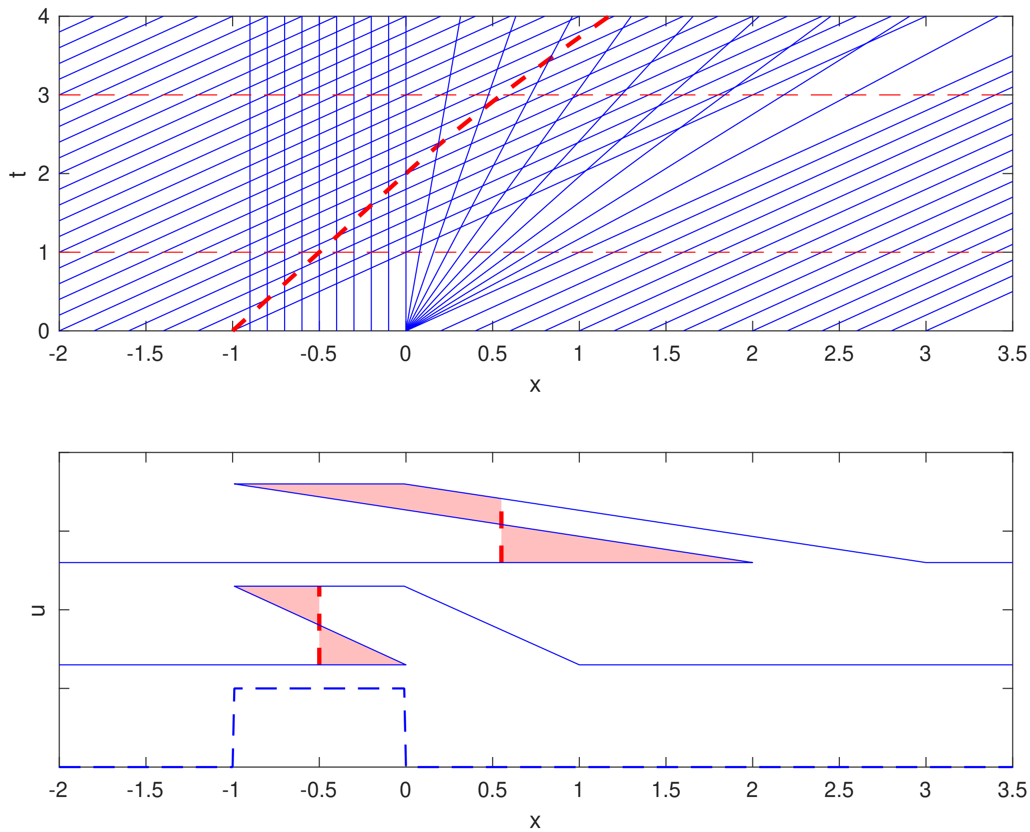
The expansion fan has the solution

$$x = (1-u)t \quad \text{or} \quad u = 1 - \frac{x}{t}, \qquad \text{for } 0 < x < t.$$

Once the shock hits the fan, $u^+ = 1 - X/t$, and so

$$\frac{dX}{dt} = \frac{1}{2} + \frac{X}{2t} \qquad \rightarrow \qquad X = t - \sqrt{2t}.$$

See the figure for more plots. The equal areas rule corresponds to the surgical removal of equal areas to either side of the vertical cut representing the shock, which guarantees that the conservation law underlying the PDE remains satisfied.



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