Coursework 1: Separation of Variables

Hand in solutions to the questions on pages 1-3 only; the later pages contain supplementary "warm-up" problems to practice on for your own enjoyment.

1. Solve the heat equation,

$$u_t = u_{xx},$$

for $0 \leq x \leq \pi$ and $t \geq 0$, subject to the boundary conditions,

$$u_x(0,t) = u_x(\pi,t) = 0,$$

and the intial condition,

$$u(x,0) = \sin 2x.$$

The MATLAB code on the next page provides a numerical solution to this problem using an in-built solver function PDEPE (output in figure 1). Compare the numerical solution with your series solution, truncated to include only the first eight terms, at the times and positions plotted in the figure. Comment on the main discrepancy between the numerical solution and your truncated analytical solution. In the third panel, the red dashed line shows the function $\frac{8}{3\pi}e^{-t}$; give a reason why this function should be close to $u(0,t)$ and $u(\frac{1}{4}\pi,t) \times \sqrt{2}$.

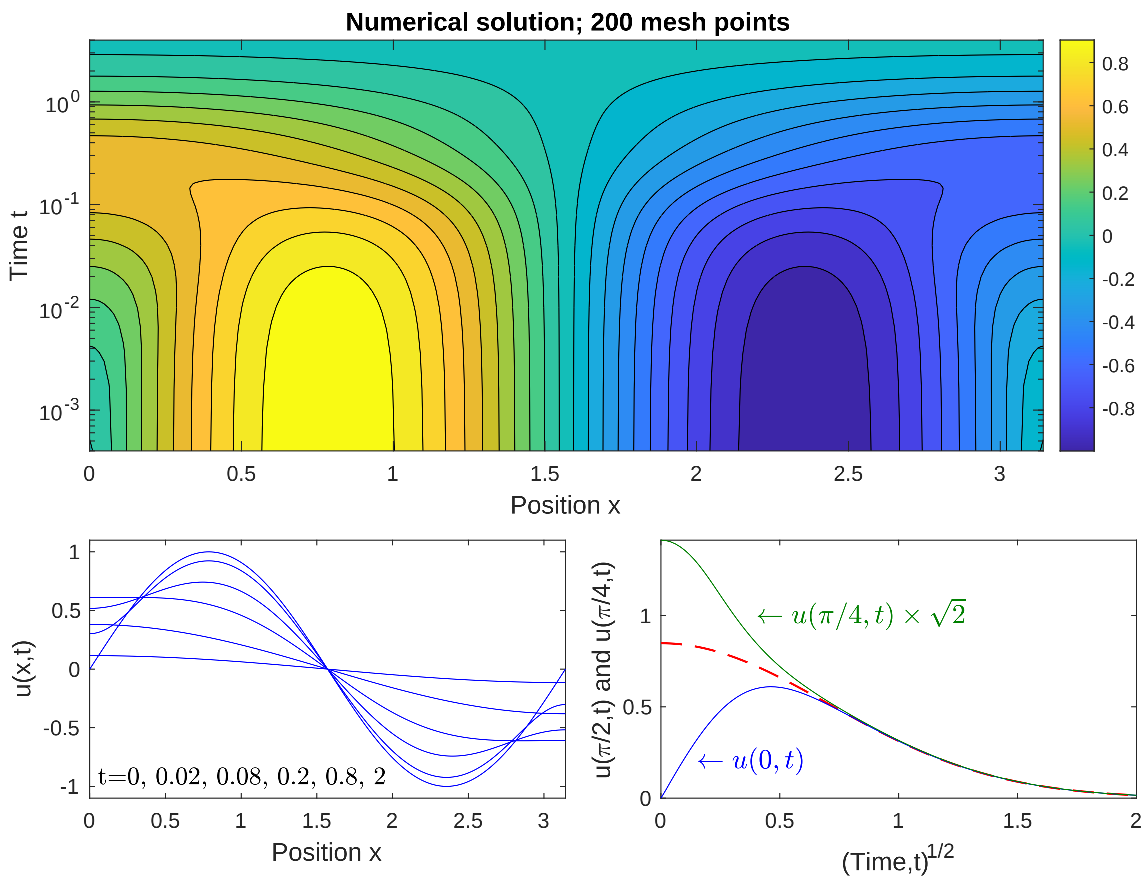

Figure 1: Numerical solution to problem 1. To show more of the early-time behaviour, the top panel plots time logarithmically, whereas the bottom right plot uses $\sqrt{t}$.

```
function pde25

        % Solves the PDE using an in-built function PDEPE

x = linspace(0,pi,200);                              % Spatial grid
t = sort([.02 .08 .2 .8 2 linspace(0,1,101).^2*4]);  % Output times
sol = pdepe(0,@pdex1pde,@pdex1ic,@pdex1bc,x,t);      % Solve PDE
u = sol(:,:,1);                                      % The solution

                % Everything else is output

subplot('position',[0.1 0.48 0.88 0.46])
contourf(x,t,u,20), colorbar
title('Numerical solution; 200 mesh points','fontsize',12)
xlabel('Position x','fontsize',12)
ylabel('Time t','fontsize',12)
set(gca,'yscal','log')

ns=[1 9 17 26 49 76]; times = t(ns)
subplot('position',[0.1 0.09 0.4 0.29])
plot(x,u(ns,:),'b');
xlabel('Position x','fontsize',12)
ylabel('u(x,t)','fontsize',12)
axis([0 pi -1.1 1.1])
text(.05,-.93,'t=0, 0.02, 0.08, 0.2, 0.8, 2','fontsize',12,...
     'interpreter','latex')

subplot('position',[0.58 0.09 0.4 0.29])
plot(t.^.5,8/3/pi*exp(-t),'r--','linewidth',1); hold on
plot(t.^.5,u(:,1),'b')
plot(t.^.5,sqrt(2)*u(:,50),'color',[0 .5 0]), hold off
xlabel('(Time,t)^{1/2}','fontsize',12)
ylabel('u(\pi/2,t) and u(\pi/4,t)','fontsize',12)
axis([0 t(end).^.5 0 inf])
text(.4,1,'$\leftarrow u(\pi/4,t)\times\sqrt2$','fontsize',13,...
     'color',[ 0 .5 0],'interpreter','latex')
text(.15,.2,'$\leftarrow u(0,t)$','fontsize',13,...
     'color','b','interpreter','latex')

function [c,f,s] = pdex1pde(x,t,u,DuDx)          % The PDE
c = 1;  f = DuDx;  s = 0;
end

function u0 = pdex1ic(x)                         % The IC
u0 = sin(2*x);
end

function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)   % The BCs
pl = 0;  ql = 1;  pr = 0;  qr = 1;
end
```

2. If $0 < a < 1$, solve the PDE

$$r^{1-2a}\frac{\partial}{\partial r}\left(r^{2a+1}\frac{\partial \phi}{\partial r}\right) + \frac{\partial^2 \phi}{\partial \theta^2} + a^2\phi = 0, \quad R < r < 1, \quad 0 \le \theta \le \Theta,$$

subject to the boundary conditions,

$$\phi(r,0) = \phi(r,\Theta) = 0, \quad \phi(R,\theta) = 0, \quad \phi(1,\theta) = f(\theta),$$

expressing your answer as a suitable sine series. Now take the limit $R \to 0$ and sum the series to leave your answer in terms of a single integral.

3. Consider the PDE:

$$u_{tt} - u_{xxt} + u_t = u_{xx} + q(x,t), \quad 0 < x < \pi.$$

The final term $q(x,t)$ is a prescribed (*i.e.* known) source. The boundary and initial conditions are

$$u(0,t) = u(\pi,t) = u_t(x,0) = 0, \qquad u(x,0) = f(x).$$

Expand u, f and q in terms of Fourier sine series (with time-dependent coefficients in the cases of $u(x,t)$ and $q(x,t)$), to find a set of ODEs with suitable initial conditions that are equivalent to the original PDE. Solve these ODEs and hence reconstruct the solution to the PDE for

$$(i) \quad q = 0, \quad f = x(\pi - x).$$

$$(ii) \quad q = tx(\pi - x), \quad f = 0,$$

and

$$(iii) \quad q = -f_{xx}, \quad f = x(\pi - x).$$

Warm-up problems

1. Solve

$$u_t = \alpha u + u_{xx}, \qquad 0 \le x \le \pi, \qquad u_x(0,t) = u_x(\pi,t) = 0,$$

where α is a constant parameter, subject to (a) $u(x,0) = \cos x$, and (b) $u(x,0) = f(x)$ for some prescribed function $f(x)$.

Using separation of variables, we have $u(x,t) = X(x)T(t)$, *with*

$$X_{xx} = -\lambda X, \qquad T_t = (\alpha - \lambda)T.$$

Applying $X_x(0) = X_x(\pi) = 0$ *implies*

$$\lambda = n^2, \qquad X = \cos nx, \qquad n = 0, 1, 2, ...$$

Hence,

$$u(x,t) = \frac{1}{2}a_0 e^{\alpha t} + \sum_{n=1}^{\infty} a_n e^{(\alpha - n^2)t} \cos nx.$$

In (a), $a_n = 0$ *for* $n \neq 1$ *and* $a_1 = 1$, *so* $u = e^{(\alpha-1)t}\cos x$. *In (b) we extend* $f(x)$ *as an even function to* $-\pi \le x \le \pi$, *allowing us to write*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \qquad a_j = \frac{2}{\pi}\int_0^{\pi} f(x)\cos jx \, dx.$$

2. Solve $\nabla^2 u = 0$ outside the unit disk, $r \ge 1$, subject to $u(1,\theta) = f(\theta)$ and u bounded as $r \to \infty$.

The solution by separation of variables is

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} r^{-m}[a_m \cos m\theta + b_m \sin m\theta],$$

after discarding the solutions r^m *and* $\ln r$ *which diverge for* $r \to \infty$. *The Fourier series expression of* $f(\theta)$ *furnishes the coefficients* a_j *and* b_j *as the usual integrals:*

$$a_j = \frac{1}{\pi}\int_0^{2\pi} f(\hat\theta)\cos j\hat\theta \, d\hat\theta, \qquad b_j = \frac{1}{\pi}\int_0^{2\pi} f(\hat\theta)\sin j\hat\theta \, d\hat\theta.$$

Introducing these integrals into the solution, using $\cos(A-B) = \cos A\cos B + \sin A \sin B$ *and* $\cos z = (e^{iz} + e^{-iz})/2$, *interchanging the order of the integral and the sum, and then summing the series using*

$$\frac{x}{(1-x)} = \sum_{m=1}^{\infty} x^m,$$

gives, with a little algebra,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2-1)f(\hat\theta)d\hat\theta}{r^2 + 1 - 2r\cos(\hat\theta - \theta)}.$$

3. If $u(x,0) = f(x)$, solve

$$u_t = \alpha u - u_{xxxx}, \qquad 0 \le x \le \pi \qquad u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0,$$

and then

$$u_t = \alpha u - u_{xxxx} + g(x), \qquad 0 \le x \le \pi \qquad u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0,$$

where α is a constant, by expanding $u(x,t)$ as a Fourier sine series with time-dependent coefficients.

For the homogeneous problem, we set

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin nx.$$

From the PDE,

$$\sum_{n=1}^{\infty} (\dot{B}_n - \alpha B_n + n^4 B_n) \sin nx = 0.$$

Hence, the coefficients satisfy the ODEs

$$\dot{B}_n = \alpha B_n - n^4 B_n.$$

The initial condition can be extended to $-\pi \le x \le \pi$ as an odd function, and $f(x)$ therefore written as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

Hence,

$$B_n(t) = b_n e^{(\alpha - n^4)t}.$$

In the inhomogeneous case, we extend $g(x)$ to $-\pi \le x \le \pi$ as an odd function:

$$g(x) = \sum_{n=1}^{\infty} g_n \sin nx.$$

The ODEs extracted from the PDE are now:

$$\dot{B}_n = \alpha B_n - n^4 B_n + g_n.$$

The solutions, subject to $B_n(0) = b_n$ are

$$B_n(t) = (b_n - G_n) e^{(\alpha - n^4)t} + G_n, \qquad G_n = \frac{g_n}{n^4 - \alpha},$$

provided $\alpha^{1/4}$ is not an integer. If $\alpha = N^4$, for some integer N, then the solution above remains valid for $n \ne N$; for $n = N$, the ODE is

$$\dot{B}_N = g_N \qquad \longrightarrow \qquad B_N = b_N + g_N t.$$

4. Solve the heat equation,

$$u_t = u_{xx},$$

for $0 \le x \le \pi$ and $t \ge 0$, subject to the boundary conditions,

$$u_x(0,t) = u_x(\pi,t) = 0,$$

and the intial condition,

$$u(x,0) = e^{-x}.$$

The solution takes the form of a Fourier cosine series:

$$u(x,t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx$$

with (using integration by parts)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} e^{-x} dx = \frac{2}{\pi}(1 - e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-x} \cos nx \; dx = \frac{2}{\pi}\frac{[1 - (-1)^n e^{-\pi}]}{n^2 + 1}.$$

5. Solve Laplace's equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0,$$

inside the annulus, $a \le r \le b$, subject to the boundary conditions,

$$(i) \quad \phi(a,\theta) = 0, \quad \phi(b,\theta) = f(\theta),$$

$$(ii) \quad \phi(a,\theta) = g(\theta), \quad \phi(b,\theta) = 0$$

and

$$(iii) \quad \phi(a,\theta) = g(\theta), \quad \phi(b,\theta) = f(\theta),$$

where $f(\theta)$ and $g(\theta)$ are two periodic functions with Fourier series representations,

$$f(\theta) = \sum_{n=1}^{\infty} f_n \sin n\theta \qquad \text{and} \qquad g(\theta) = \frac{1}{2}g_0 + \sum_{n=1}^{\infty} g_n \cos n\theta.$$

The solutions are

$$(i) \quad \phi(r,\theta) = \sum_{n=1}^{\infty} f_n \sin n\theta \frac{(r^n - a^{2n} r^{-n})}{(b^n - a^{2n} b^{-n})}$$

$$(ii) \quad \phi(r,\theta) = \tfrac{1}{2}g_0 \frac{\ln(r/b)}{\ln(a/b)} + \sum_{n=1}^{\infty} g_n \cos n\theta \frac{(r^n - b^{2n} r^{-n})}{(a^n - b^{2n} a^{-n})}$$

The solution to (iii) *is just the sum of the two solutions above.*

6. Consider the heat equation with a source term, $q(x,t)$:

$$u_t = u_{xx} + q.$$

The boundary and initial conditions are

$$u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x).$$

Expand u, f and q in terms of Fourier sine series (with time-dependent coefficients in the cases of $u(x,t)$ and $q(x,t)$), to find a set of ODEs with suitable initial conditions that are equivalent to the original PDE. Solve these ODEs and hence reconstruct the solution to the PDE for

$$(i) \quad q = 0$$

and

$$(ii) \quad q = e^{-t} f(x).$$

First, we set

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

(with a bunch of computable constants b_n, given $f(x)$),

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right) \qquad \& \qquad q(x,t) = \sum_{n=1}^{\infty} Q_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

These allow the PDE to be broken down into the system of ODEs,

$$\dot{B}_n + \frac{n^2\pi^2}{L^2} B_n = Q_n,$$

subject to the initial conditions $B_n(0) = b_n$. The solutions are

$$(i) \qquad B_n = \frac{L^2 b_n}{n^2\pi^2} e^{-n^2\pi^2 t/L^2}, \qquad (Q_n = 0),$$

$$(ii) \qquad B_n = \frac{L^2 b_n}{n^2\pi^2 - 1}(e^{-t} - e^{-n^2\pi^2 t/L^2}) + \frac{L^2 b_n}{n^2\pi^2} e^{-n^2\pi^2 t/L^2}, \qquad (Q_n = b_n e^{-t}),$$

with $B_n(t) = (1+t) b_n e^{-t}$ if $n\pi = L$.

Actual solutions

1. *(9 points)* After separation of variables, the solution takes the form of a Fourier cosine series:

$$u(x,t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx$$

with

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx = 0, \quad a_n = \frac{1}{\pi}\int_0^{\pi} \sin 2x \cos nx \, dx = \frac{4[1-(-1)^n]}{\pi(4-n^2)} \quad (a_2 = 0).$$

(4 points for the solution and coefficients).

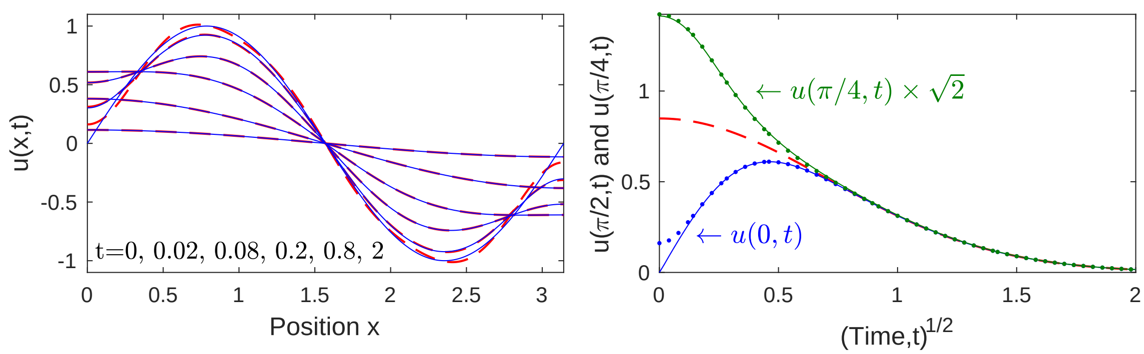

Figure 2: Dashed lines and dots are from the truncated series.

The figure above compares the truncated sum (including eight terms, so upto and including $n = 7$, given the first term is $\frac{1}{2}a_0$) with the numerical solution. The main discrepancies between the truncated analytical solution and the numerical one arise for $t = 0$, where the Fourier series is attempting to represent a function with a discontinuous derivative at $x = 0$ and π. For small times, the discontinuous derivative is smoothed out, but the truncation of the series remains inaccurate. The reason that $\sqrt{2}\, u(\frac{1}{4}\pi, t)$ and $u(0,t)$ are close to the particular exponential function for later times is that the first term of the Fourier series solution $a_1 e^{-t}\cos x$ dominates the other terms for $t > 1$ and $a_1 = \frac{8}{3\pi}$. (5 points; 3 for the pictures, 1 for rationalizing the discrepancy and 1 for explaining the decay rate)

2. *(9 points)* Separating variables, $\phi(r,\theta) = R(r)\Theta(\theta)$, gives

$$\Theta'' + m^2\Theta = 0 \qquad \& \qquad r^{1-2a}(r^{2a+1}R')' + (a^2 - m^2)R = 0$$

for some separation constant m^2. Hence $\Theta \propto \sin m\theta$, $m = n\pi/\Theta$ and $n = 1, 2, ...$; the $R-$equation takes the Euler form, with solutions $R \propto r^{\pm m - a}$ (3 points). Hence,

$$\phi(r,\theta) = \frac{1}{r^a}\sum_{n=1}^{\infty} f_n \sin m\theta \frac{(r^m - R^{2m}r^{-m})}{(1-R^{2m})}. \quad \text{(2 points)}$$

For $R \to 0$, we have

$$\phi = \frac{1}{r^a}\sum_{n=1}^{\infty} (r^q)^n f_n \sin nq\theta, \quad q = \frac{\pi}{\Theta}.$$

Using the same manipulations as for warm-up problem 2, we see that

$$\phi(r,\theta) = \frac{1}{\Theta r^a}\int_0^{\Theta}\left[\frac{r^q\cos q(\theta-\hat{\theta}) - r^{2q}}{r^{2q}+1-2r^q\cos q(\theta-\hat{\theta})} - \frac{r^q\cos q(\theta+\hat{\theta}) - r^{2q}}{r^{2q}+1-2r^q\cos q(\theta+\hat{\theta})}\right] f(\hat{\theta})d\hat{\theta}.$$

(4 points).

3. *(9 points)* First,

$$x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with $b_n = 0$ for n even and

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\, dx = \frac{8}{\pi n^3}$$

for n odd (2 points).

Now set

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin nx \qquad \& \qquad q(x,t) = \sum_{n=1}^{\infty} Q_n(t)\sin nx,$$

allowing the PDE to be broken down into the system of ODEs,

$$\ddot{B}_n + (n^2+1)\dot{B}_n + n^2 B_n = Q_n,$$

subject to the initial conditions $\dot{B}_n(0) = 0$, and $B_n(0) = b_n$ in parts (i) and (iii), then $B_n(0) = 0$ in part (ii) (2 points).

The solutions are

$$(i) \qquad B_n = \frac{b_n}{n^2-1}(n^2 e^{-t} - e^{-n^2 t}) \quad \& \quad B_1 = b_1(1+t)e^{-t}, \qquad (Q_n = 0),$$

$$(ii) \qquad B_n = \frac{b_n}{n^4}\left[n^2 t - n^2 - 1 + \frac{n^4(1+n^2)e^{-t} - (1+n^4)e^{-n^2 t}}{n^2-1}\right]$$

or

$$B_1 = b_1[(3+2t)e^{-t} + t - 2], \qquad (Q_n = t b_n),$$

$$(iii) \qquad B_n = \frac{b_n}{n^2} \qquad (Q_n = b_n).$$

(5 points)