Coursework 2: Sturm-Liouville problems and Bessel functions

Hand in solutions to the questions on page 1 only; later pages contain helpful information and supplementary "warm-up" problems to practice on for your own enjoyment. Be as explicit as you can in providing your answers.

(1). Consider a uniformly heated disk, with temperature $u(r, \theta, t)$ satisfying

$$u_t = \nabla^2 u + 4 = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + 4, \quad u(1, \theta, t) = 0, \quad u(r, \theta, 0) = \cos\theta.$$

First, find the steady-state solution, $u = U(r)$, which satisfies

$$\nabla^2 U + 4 = 0 \quad \& \quad U(1) = 0.$$

Now set $u(r, \theta, t) = U(r) + v(r, \theta, t)$ and solve for $v(r, \theta, t)$ using separation of variables. Compare your result, with the series truncated to six terms, with a numerical solution based on the MATLAB code provided below, at the times and radii indicated in the figure. In a sentence or two, assess the performance of your truncated series.

(2). A model of a star with a nuclear burning core has a temperature given by

$$u_t = \frac{1}{r^2}(r^2 u_r)_r + \frac{3u}{16r^2}, \quad u(1, t) = 0, \quad u(r, 0) = 1,$$

and $\sqrt{r}u \to 0$ for $r \to 0$. Provide an expression for the star's luminosity, $L = -u_r(1, t)$. Hence establish that the star slowly burns out and give an approximate formula for its half-life (the time required for L to reduce by a factor of a half).

Helpful notes: Bessel's equation is

$$x^2 y'' + xy' + (k^2 x^2 - \nu^2)y = 0,$$

and has the two solutions, $y = J_\nu(kx)$ and $Y_\nu(kx)$, of which only the former is regular at $z = 0$. For $z \to 0$, $J_\nu(z) \propto z^\nu$.

The more general ODE,

$$x^2 y'' + (1 - 2\alpha)xy' + (\omega^2\beta^2 x^{2\beta} + \alpha^2 - \nu^2\beta^2)y = 0,$$

has solutions $y = x^\alpha \mathcal{C}_\nu(\omega x^\beta)$ where $\mathcal{C}_\nu(z)$ is a Bessel function.

If ν is equal to an integer m, the Bessel functions satisfy the recurrence relation,

$$J_{m-1}(z) - J_{m+1}(z) = 2J'_m(z), \qquad J'_0(z) = -J_1(z).$$

Remember, Bessel functions are our friends.

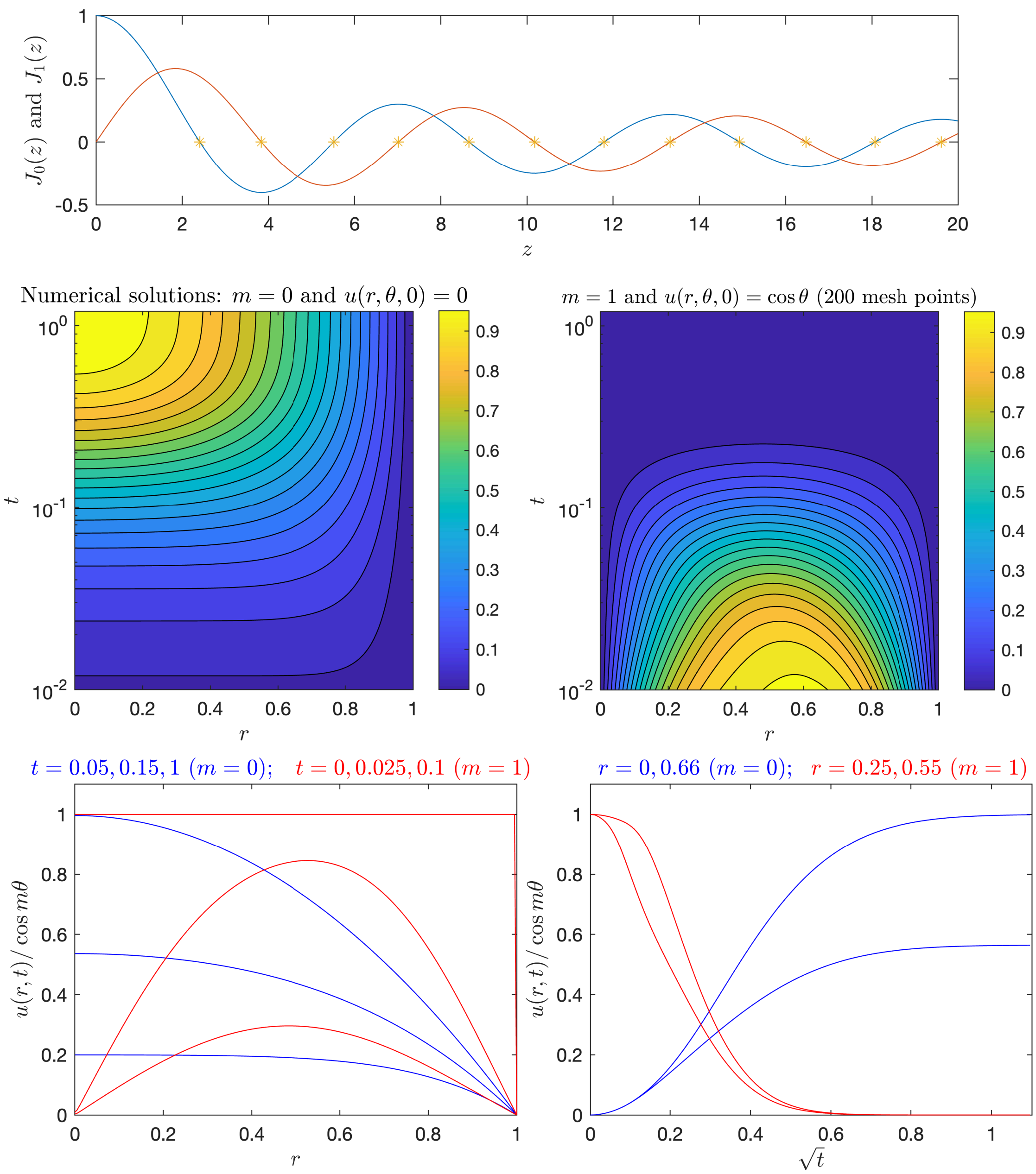

Figure 1: Top panel: the Bessel functions $J_0(z)$ and $J_1(z)$ and their zeros. Middle panels: numerical solutions to $u_t = \nabla^2 u + 4(1 - m)$ and $u(1, \theta, t) = 0$, for $m = 0$ and $u(r, \theta, 0) = 0$ (left) and $m = 1$ and $u(r, \theta, 0) = \cos\theta$ (right). The lower panels show snapshots and time series of the two solutions at the times and positions indicated.

```
function pd25b

' Solves the PDE using an in-built function PDEPE'

r = linspace(0,1,201);    % spatial grid for r
z = linspace(0,20,201);   % spatial grid for z
tp0 = [0.05 0.15 1]; tp1 = [0 0.025 0.1];  % specific output times
t = sort(unique([1.2*linspace(0,1,200).^2 tp0 tp1]));
ra = linspace(0,1,40);    % another spatial grid
ntrunc=6;

close all, subplot('position',[0.1 0.83 0.82 0.16])

% Calculate Bessel functions, find some zeros, and plot it all
zn = [];
for nu = [0 1];
    for n=1:ntrunc
        zng=(n*2-1)*pi/2+pi/2*nu+pi/4;             % guesses
        zn = [zn fzero(@(z) besselj(nu,z),zng)];   % compute zeros
    end
end
zn
plot(z,besselj(0,z),z,besselj(1,z),zn,0*zn,'*')
xlabel('$z$','fontsize',12,'interpreter','latex')
ylabel('$J_0(z)$ and $J_1(z)$','interpreter','latex','fontsize',12)

%  Compute some integrals involving a Bessel function:
J=besselj(1,r*zn(7)); I1=trapz(r,r.*J), I2=trapz(r,r.*J.^2)

%  Solve PDE, first for m=0, then for m=1,
m = 0; u0 = pdepe(1,@pdex1pde,@pdex1ic,@pdex1bc,r,t);
m = 1; u1 = pdepe(1,@pdex1pde,@pdex1ic,@pdex1bc,r,t);

% Plot results
subplot('position',[0.08 0.42 0.41 0.32])
contourf(r,t,u0,20),
set(gca,'yscal','log'), ylim([1e-2 1.2])
title('Numerical solutions: $m=0$ and $u(r,\theta,0)=0$',...
     'interpreter','latex','fontsize',12)
xlabel('$r$','interpreter','latex','fontsize',12)
ylabel('$t$','interpreter','latex','fontsize',12)
colorbar('Ticks',[0:.1:1])
subplot('position',[0.58 0.42 0.41 0.32])
contourf(r,t,u1,20),
set(gca,'yscal','log'), ylim([1e-2 1.2])
title('$m=1$ and $u(r,\theta,0)=\cos\theta$ (200 mesh points)',...
     'interpreter','latex')
xlabel('$r$','interpreter','latex','fontsize',12)
ylabel('$t$','interpreter','latex','fontsize',12)
colorbar('Ticks',[0:.1:1])

ns0=[]; for tt = tp0, ns0 = [ns0 find(t==tt)]; end
ns1=[]; for tt = tp1, ns1 = [ns1 find(t==tt)]; end
subplot('position',[0.08 0.06 0.42 0.28])
plot(r,u0(ns0,:),'b',r,u1(ns1,:),'r')
axis([0 1 0 1.1])
text(-.1,1.15,'$t=0.05,0.15,1$ ($m=0$);',...
     'interpreter','latex','fontsize',12,'color','b')
text(.5,1.15,'$t=0,0.025,0.1$ ($m=1$)',...
     'interpreter','latex','fontsize',12,'color','r')
xlabel('$r$','interpreter','latex','fontsize',12)
ylabel('$u(r,t)/\cos m\theta$','interpreter','latex','fontsize',12)

subplot('position',[0.57 0.06 0.42 0.28])
plot(sqrt(t),u0(:,[1 133]),'b'), hold on
plot(sqrt(t),u1(:,[51 111]),'r'), hold off
xlabel('$\sqrt t$','interpreter','latex','fontsize',12)
ylabel('$u(r,t)/\cos m\theta$','interpreter','latex','fontsize',12)
axis([0 1.1 0 1.1])
text(0.02,1.15,'$r=0,0.66$ ($m=0$);',...
     'interpreter','latex','fontsize',12,'color','b')
text(.55,1.15,'$r=0.25,0.55$ ($m=1$)',...
     'interpreter','latex','fontsize',12,'color','r')

figgy = gcf; figgy.Position(4)=3/2*figgy.Position(4);

function [c,f,s] = pdex1pde(r,t,u,DuDr)
c = 1;  f = DuDr;  s = -m^2*u/r^2+4*(1-m);
end

function u0 = pdex1ic(r)
  u0 = m;
end

function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
pl = m*ul;  ql = 1-m;  pr = ur;    qr = 0;
end

end
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Warm-up problems

(1). The equation of motion of a hanging, heavy chain is

$$u_{tt} = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right),$$

where $u(x,t)$ is the horizontal deflection at height x and time t (the tension in the chain varies with height due to the weight underneath). The end at $x = 0$ is free, whereas the end at $x = l$ is fixed, so that u is regular for $x = 0$ and $u(l,t) = 0$. Using separation of variables reduce the PDE to two equivalent ODEs. Show that the spatial dependence of the solution is given by the Bessel function, $J_0(z)$. *Hint: the transformation $x = cz^2$ may prove helpful, for some constant c.*

Given that the zeros of $J_0(z)$ are $z = z_1, z_2, ..., z_n, ...$, write down a general solution of the PDE in terms of a sum over Bessel functions with unspecified coefficients. If $u(x,0) = 0$ and $u_t(x,0) = f(x)$, express those coefficients in terms of integrals of $J_0(z)$.

Separation of variables: $u = X(x)T(t)$, with

$$xX'' + X' + \lambda_n^2 X = 0, \qquad T = a_n \cos\lambda_n t + b_n \sin\lambda_n t.$$

Making the suggested change of variable and choosing $c = 1/(4\lambda_n^2)$, leads to

$$X_{zz} + \frac{1}{z}X_z + X = 0 \quad \longrightarrow \quad X = J_0(z) = J_0(2\lambda_n\sqrt{x}),$$

on using the regularity of $J_0(z)$ at $z = 0$. The other boundary condition implies that $\lambda_n = z_n/2\sqrt{l}$, where z_n is the n^{th} zero of $J_0(z)$. Thus,

$$u = \sum_{n=1}^{\infty}(a_n \cos\lambda_n t + b_n \sin\lambda_n t)J_0(2\lambda_n\sqrt{x}),$$

With the given initial condition, $a_n = 0$ and b_n must be computed from a suitable expansion in Bessel functions. Given that the equation for $X(x)$ is a Sturm-Liouville problem with weight $\sigma(x) = 1$, the J_0's form an orthogonal basis set, and we arrive at

$$b_n = \frac{2\sqrt{l}}{z_n}\frac{\int_0^l f(x)J_0(z_n\sqrt{x/l})dx}{\int_0^l J_0^2(z_n\sqrt{x/l})dx}.$$

(2). Using the method of separation of variables, solve Laplace's equation inside the cylinder, $0 \le r \le R$, $0 \le \theta \le 2\pi$, $0 \le z \le L$, in cylindrical polar coordinates (r,θ,z), applying the boundary condition, $u(R,\theta,z) = 0$, $u(r,\theta,0) = 0$ and

$$u(r,\theta,L) = F(r,\theta) = \frac{1}{2}F_0(r) + \sum_{m=1}^{\infty} F_m(r)\cos m\theta$$

expressing your result in terms of Bessel functions (including any constants of integration).

The PDE to solve is

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0.$$

We put $u = X(r)Y(\theta)Z(z)$ and rewrite the PDE as

$$\frac{1}{rX}(rX_r)_r + \frac{1}{r^2 Y}Y_{\theta\theta} = -\frac{Z_{zz}}{Z}.$$

The right-hand side is a function of z alone, whereas the left-hand side is a function of r and θ, so both must equal a separation constant, $-k^2$. Hence

$$\frac{r}{X}(rX_r)_r + r^2k^2 = -\frac{Y_{\theta\theta}}{Y}.$$

The right-hand side is now a function of θ, the left is a function of r; we put both equal the separation constant m^2. Consequently,

$$Z_{zz} = k^2 Z, \qquad \text{and} \qquad Y_{\theta\theta} = -m^2 Y,$$

Thus,

$$Z = \sinh kz, \qquad Y = \cos m\theta \text{ or } \sin m\theta \text{ with } m = 1, 2, ..., \text{ or } \textit{constant} \text{ with } m = 0$$

(since $Z(0) = 0$ and $Y(\theta)$ must be $2\pi-$periodic). Finally,

$$(rX_r)_r - \frac{m^2}{r}X + rk^2X = 0,$$

which, along with the boundary conditions, $X(r)$ regular at $r = 0$ and $X(R) = 0$, determine a Sturm-Liouville problem for $X(r)$ and eigenvalue k^2, with $p(r) = r$, $q(r) = -m^2/r$ and $\sigma(r) = r$. For each m, there is an infinite number of solutions, $k = k_{m,n}$ and $X(r) = X_{m,n}(r)$, $n = 1, 2, ...$ Likewise, there are similar solutions for $m = 0$. The general solution of the PDE is therefore

$$u(r,\theta,z) = \sum_{n=1}^{\infty}\left[\frac{1}{2}a_{0,n}X_{0,n}(r)\sinh(k_{0,n}z) + \sum_{m=1}^{\infty}(a_{m,n}\cos m\theta + b_{m,n}\sin m\theta)X_{m,n}(r)\sinh(k_{m,n}z)\right].$$

Comparing the ODE of the Sturm-Liouville problem with Bessel's equation, we see that

$$X_{m,n}(r) \equiv J_m(k_{m,n}r) \qquad \text{and} \qquad k_{m,n} = \frac{z_{m,n}}{R}$$

where $z_{m,n}$ is the n^{th} zero of $J_m(z)$, and the set of functions can be extended to include $m = 0$. Given also the boundary condition at $z = L$ (a cosine series in θ), we have $b_{m,n} = 0$. Finally,

$$F_m(r) = \sum_{n=1}^{\infty} a_{m,n}J_m(k_{m,n}r)\sinh(k_{m,n}L),$$

with $m = 0$, 1, 2, ..., and so

$$a_{m,n} = \frac{\int_0^R F_m(r)J_m(k_{m,n}r)rdr}{\sinh(k_{m,n}L)\int_0^R[J_m(k_{m,n}r)]^2rdr}.$$

(3). Using the method of separation of variables, solve the heat equation inside the unit disk, $r \le 1$, applying the boundary condition, $u(1,\theta,t) = 0$, and initial condition,

$$u(r,\theta,0) = \sum_{m=1}^{\infty} f_m(r)\sin m\theta.$$

expressing your result in terms of Bessel functions and their integrals.

The PDE to solve is (if one includes the diffusivity for completeness but not necessity)

$$\frac{1}{\kappa}u_t = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}.$$

We put $u = X(r)Y(\theta)T(t)$ *and rewrite the PDE as*

$$\frac{1}{rX}(rX_r)_r + \frac{1}{r^2Y}Y_{\theta\theta} = \frac{T_t}{\kappa T}.$$

The right-hand side is a function of t *alone, whereas the left-hand side is a function of* r *and* θ*, so both equal a separation constant,* $-k^2$*. Hence*

$$\frac{r}{X}(rX_r)_r + r^2k^2 = -\frac{Y_{\theta\theta}}{Y}.$$

The right-hand side is now a function of θ*, the left is a function of* r*; we put both equal the separation constant* m^2*. Consequently,*

$$T_t = -\kappa k^2 T, \qquad \text{and} \qquad Y_{\theta\theta} = -m^2 Y,$$

Thus,

$$T = Ce^{-\kappa k^2 t}, \qquad Y = A\cos m\theta \text{ or } B\sin m\theta,\ m = 0, 1, 2, ...$$

since $Y(\theta)$ *must be* $2\pi-$*periodic. Finally,*

$$(rX_r)_r - \frac{m^2}{r}X + rk^2X = 0,$$

which, along with the boundary conditions, $X(r)$ *regular at* $r = 0$ *and* $X(1) = 0$*, determine a Sturm-Liouville problem for* $X(r)$ *and eigenvalue* k^2*, with* $p(r) = r$*,* $q(r) = -m^2/r$ *and* $\sigma(r) = r$*. For each* m*, there is an infinite number of solutions,* $k = k_{m,n}$ *and* $X(r) = X_{m,n}(r)$*,* $n = 1, 2, ...$ *Comparing the ODE of the Sturm-Liouville problem with Bessel's equation, we see that*

$$X_{m,n}(r) \equiv J_m(k_{m,n}r) \qquad \text{and} \qquad k_{m,n} = z_{m,n}$$

where $z_{m,n}$ *is the* n^{th} *zero of* $J_m(z)$*. The general solution of the PDE is therefore*

$$u(r, \theta, z) = \sum_{n=1}^{\infty}\left[\frac{1}{2}a_{0,n}J_0(z_{0,n}r)e^{-\kappa z_{0,n}^2 t} + \sum_{m=1}^{\infty}(a_{m,n}\cos m\theta + b_{m,n}\sin m\theta)J_m(r)e^{-\kappa z_{m,n}^2 t}\right].$$

Finally, we observe that $u(r, \theta, 0)$ *is a sine series in* θ*, so* $a_{0,n} = a_{m,n} = 0$*, and demanding* $u(r, \theta, 0) = f(r, \theta) = \sum_m f_m(r)\sin m\theta$*, implies*

$$f_m(r) = \sum_{n=1}^{\infty} b_{m,n}J_m(z_{m,n}r)$$

and so (from the SL expansion formulae)

$$b_{m,n} = \frac{\int_0^1 f_m(r)J_m(z_{m,n}r)rdr}{\int_0^1 [J_m(z_{m,n}r)]^2 rdr}.$$

Two More...

(1). Consider the axisymmetric heat equation,

$$u_t = \frac{1}{r}(ru_r)_r$$

in $r \le R$, subject to $u(R,t) = 0$ and $u(r,t)$ regular at the origin. Determine the Sturm-Liouville (SL) problem satisfied by the radial part of the separable solution, $u(r,t) = X(r)T(t)$, establishing the form of the functions $p(r)$, $q(r)$ and $\sigma(r)$ in the ODE and stating the boundary conditions and how the eigenvalue is related to the separation constant of the PDE. Show that the eigenfunctions of the SL problem are Bessel functions, and write the eigenvalue in terms of the zeros of $J_0(z)$. Given $u(r,0) = f(r)$, express the solution to the PDE in terms of Bessel functions and their integrals.

(2). Using the method of separation of variables, solve the wave equation inside the unit disk, $r \le 1$, applying the boundary condition, $u(1,\theta,t) = 0$, and initial conditions,

$$u(r,\theta,0) = \frac{1}{2}f_0(r) + \sum_{m=1}^{\infty} f_m(r)\cos m\theta \qquad \text{and} \qquad u_t(r,\theta,0) = 0,$$

expressing your result in terms of Bessel functions and their integrals.

Solutions:

(1). Separate variables: $u = X(r)T(t)$, giving

$$\frac{1}{rX}(rX_r)_r = \frac{T_t}{T} = -k^2,$$

where $-k^2$ is the separation constant. Hence

$$(rX_r)_r + k^2 rX = 0 \qquad \text{and} \qquad T = e^{-k^2 t}.$$

The first equation is the ODE of a Sturm-Liouville (SL) problem with $p(r) = \sigma(r) = r$, $q(r) = 0$ and eigenvalue k^2. Comparison with Bessel's equation and imposition of $X(R) = 0$ indicates that

$$X(r) = J_0(kr) \qquad \text{and} \qquad J_0(kR) = 0.$$

Denoting z_n as the n^{th} zero of $J_0(z)$, $n = 1, 2, ...$, we find the SL eigenvalues, $k_n = z_n/R$, and eigenfunctions, $X_n(r) = J_0(k_n r)$. Hence,

$$u(r,t) = \sum_{n=1}^{\infty} c_n e^{-k_n^2 t} J_0(k_n r).$$

Finally, we apply the initial condition:

$$f(r) = \sum_{n=1}^{\infty} c_n J_0(k_n r) \qquad \longrightarrow \qquad c_n = \frac{\int_0^R f(r) J_0(k_n r) r dr}{\int_0^R [J_0(k_n r)]^2 r dr}.$$

(2). The PDE to solve is

$$u_{tt} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}.$$

We put $u = X(r)Y(\theta)T(t)$ and rewrite the PDE as

$$\frac{1}{rX}(rX_r)_r + \frac{1}{r^2Y}Y_{\theta\theta} = \frac{T_{tt}}{T}.$$

The right-hand side is a function of t alone, whereas the left-hand side is a function of r and θ, so both equal a separation constant, $-k^2$. Hence

$$\frac{r}{X}(rX_r)_r + r^2k^2 = -\frac{Y_{\theta\theta}}{Y}.$$

The right-hand side is now a function of θ, the left is a function of r; we put both equal the separation constant m^2. Consequently,

$$T_{tt} = -k^2T, \qquad \text{and} \qquad Y_{\theta\theta} = -m^2Y,$$

Thus,

$$T = \cos kt, \qquad Y = \cos m\theta \text{ or } \sin m\theta,\ m = 0, 1, 2, ...$$

since $u_t(r, \theta, 0) = 0$ (or $T_t(0) = 0$) and $Y(\theta)$ must be $2\pi-$periodic. Finally,

$$(rX_r)_r - \frac{m^2}{r}X + rk^2X = 0,$$

which, along with the boundary conditions, $X(r)$ regular at $r = 0$ and $X(1) = 0$, determine a Sturm-Liouville problem for $X(r)$ and eigenvalue k^2, with $p(r) = r$, $q(r) = -m^2/r$ and $\sigma(r) = r$. For each m, there is an infinite number of solutions, $k = k_{m,n}$ and $X(r) = X_{m,n}(r)$, $n = 1, 2, ...$ The general solution of the PDE is therefore

$$u(r, \theta, z) = \sum_{n=1}^{\infty}\left[\frac{1}{2}a_{0,n}X_{0,n}(r)\cos(k_{0,n}t) + \sum_{m=1}^{\infty}(a_{m,n}\cos m\theta + b_{m,n}\sin m\theta)X_{m,n}(r)\cos(k_{m,n}t)\right].$$

Comparing the ODE of the Sturm-Liouville problem with Bessel's equation, we see that

$$X_{m,n}(r) \equiv J_m(k_{m,n}r) \qquad \text{and} \qquad k_{m,n} = z_{m,n}$$

where $z_{m,n}$ is the n^{th} zero of $J_m(z)$.

Finally, we observe that $u(r, \theta, 0)$ is a cosine series in θ, so $b_{m,n} = 0$, and

$$u(r, \theta, z) = \sum_{n=1}^{\infty}\left[\frac{1}{2}a_{0,n}J_0(k_{0,n}r)\cos(k_{0,n}t) + \sum_{m=1}^{\infty}a_{m,n}\cos m\theta\ J_m(k_{m,n}r)\cos(k_{m,n}t)\right].$$

Finally, demanding $u(r, \theta, 0) = f(r, \theta) = \frac{1}{2}f_0(r) + \sum_m f_m(r)\cos m\theta$, implies

$$f_m(r) = \sum_{n=1}^{\infty}a_{m,n}J_m(k_{m,n}r)$$

(for $m = 0, 1, 2...$), and so

$$a_{m,n} = \frac{\int_0^R f_m(r)J_m(k_{m,n}r)rdr}{\int_0^R[J_m(k_{m,n}r)]^2rdr}.$$

A previous year's assignment:

(1). The temperature $T(r,\theta,t)$ in a heated circular swimming pool satisfies

$$T_t = \frac{1}{r}(rT_r)_r + \frac{1}{r^2}T_{\theta\theta} + \alpha, \qquad T(1,\theta,t) = 0,$$

where the heating rate α is a prescribed constant. First, find the temperature distribution $T(r,\theta,t) = T_{ss}(r)$ if the pool were in steady state. Next, by putting $T(r,\theta,t) = T_{ss}(r) + u(r,\theta,t)$, solve the PDE for $u(r,\theta,t)$ using separation of variables, imposing the initial condition,

$$T(r,\theta,0) = f(r)\sin 2\theta,$$

and expressing your result in terms of Bessel functions and their integrals.

(2). Consider the PDE,

$$u_t = (ru_r)_r + \frac{1}{r}u_{\theta\theta}$$

in $r \le 1$, subject to $u(1,\theta,t) = 0$ and the conditions that $u(r,\theta,t)$ is $2\pi-$periodic in θ and regular at $r = 0$.

(a) Determine the Sturm-Liouville (SL) problem satisfied by the radial part of the separable solution, $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, establishing the form of the functions $p(r)$, $q(r)$ and $\sigma(r)$ in the ODE, and stating the boundary conditions and how the eigenvalue is related to the separation constant of the PDE. Show that the eigenfunctions of the SL problem are Bessel functions, and write the eigenvalue in terms of the zeros of a Bessel function. Given $u(r,\theta,0) = f(r,\theta)$, express the solution to the PDE in terms of Bessel functions and their integrals.

(b) As shown in figure 2, the numerical solution to the axisymmetric problem, with $u = u(r,t)$ and $f(r) = 16r^2(1-r)^2$, eventually decays exponentially at each radial position, with a rate 1.45. Explain this observation.

(c) If $f(r,\theta) = 16r^2(1-r)^2\sin\theta$, write down a reduced version of your separation of variables solution. Compute the coefficients for the first five terms of the series, then compare your results with the numerical solution shown in figure 3, at the times and positions indicated in the lowest two panels.

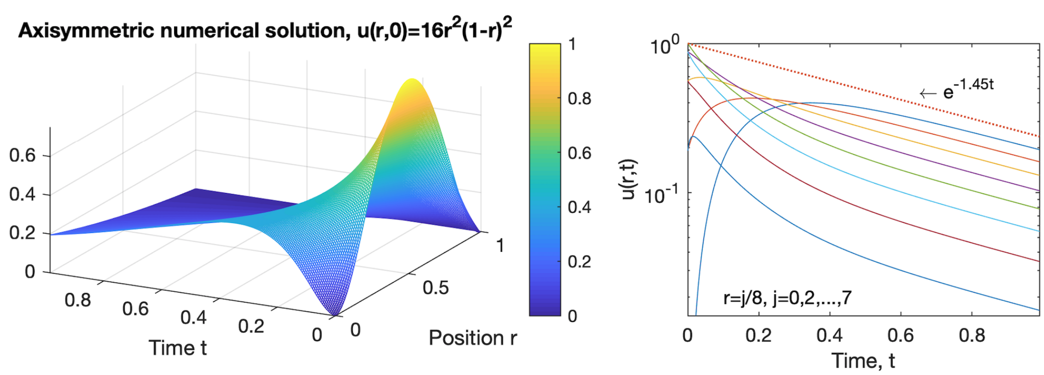

Figure 2: Axisymmetric numerical solution

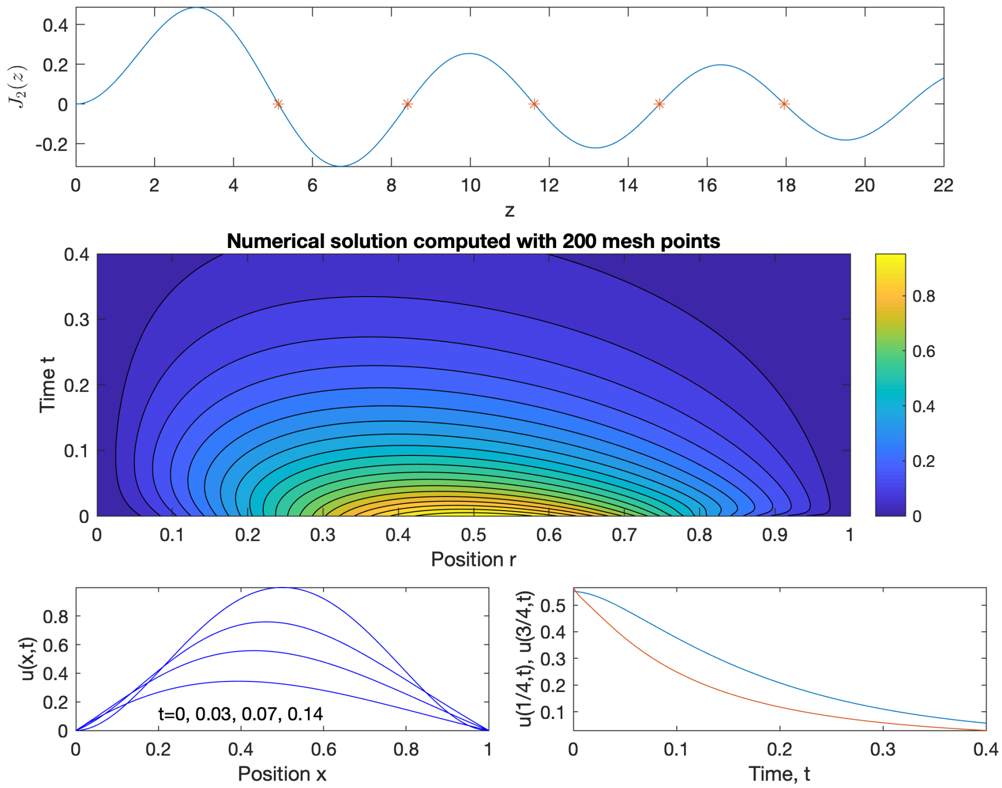

Figure 3: Output from pde21b.m.

Solutions

1. We have

$$(rT'_{ss})' + \alpha r = 0 \quad \rightarrow \quad T'_{ss} + \frac{1}{2}\alpha r = 0 \quad \rightarrow \quad T_{ss} = \frac{1}{4}\alpha(1 - r^2)$$

(avoiding any singularities at $r = 0$ and since $T_{ss}(1) = 0$). If $T(r, t) = T_{ss}(r) + u(r, t)$, then $u(r, t)$ satisfies

$$u_t = \frac{1}{r}(ru_r)_r, \quad u(1, \theta, t) = 0, \quad u(r, \theta, 0) = f(r) \sin 2\theta - T_{ss}(r).$$

We separate variables, $u = R(r)\Theta(\theta)T(t)$, giving the ODEs

$$T' + \lambda T = 0, \qquad \Theta'' + m^2\Theta = 0, \qquad (rR')' + \lambda rR - m^2R = 0,$$

for two separation constants λ and m. We choose $m = 0$ and $\Theta = \frac{1}{2}A_0$, or $m = 1, 2, ...$ and $\Theta = B_m \sin m\theta$ or $A_m \cos m\theta$ to guarantee 2π-periodic solutions in θ, in the usual manner of a Fourier series. In fact, the initial condition indicates that we only need the $(m, \Theta) = (0, \frac{1}{2}A_0)$ and $(m, \Theta) = (2, B_2 \sin 2\theta)$ solution pairs. The ODE for $R(r)$ is Bessel's equation, with either $J_0(kr)$ or $J_2(kr)$ as solutions, given that $m = 0$ or 2, with $\lambda = k^2$. But $u(1, \theta, t) = 0$ implies that $R(1) = 0$ and so k must be a zero of the corresponding Bessel function. *i.e.* $\lambda = k_{0,n}^2$ for $m = 0$, or $\lambda = k_{2,n}^2$ for $m = 2$, with $J_m(k_{m,n}) = 0$ and $n = 1, 2, ...$ Altogether, we find the general solution,

$$u(r, \theta, t) = \sum_{n=1}^{\infty} \left[a_n J_0(k_{0,n}r)e^{-k_{0,n}^2 t} + b_n J_2(k_{2,n}r)e^{-k_{2,n}^2 t} \sin 2\theta \right],$$

for a suitable set of constants a_n and b_n. Last, in view of the initial condition and the Sturm-Liouville expansion theorem, we see that

$$a_n = -\frac{1}{4}\alpha \int_0^1 (1-r^2) J_0(k_{0,n} r)\ r dr \left[\int_0^1 [J_0(k_{0,n} r)]^2\ r dr\right]^{-1}$$

and

$$b_n = \int_0^1 f(r) J_2(k_{2,n} r)\ r dr \left[\int_0^1 [J_2(k_{2,n} r)]^2\ r dr\right]^{-1}.$$

2(a) Separating variables, we arrive at the ODEs

$$T' + \lambda T = 0, \qquad \Theta'' + m^2 \Theta = 0, \qquad (rR')' + \lambda R - \frac{m^2}{r} R = 0,$$

for two separation constants λ and m. We choose $m = 0$ and $\Theta = \textit{constant}$, or $m = 1, 2, ...$ and $\Theta \propto \sin m\theta$ or $\cos m\theta$ to guarantee 2π-periodic solutions in θ. The ODE for the $r-$dependence is a Sturm-Liouville problem with $p \equiv r$, $\sigma \equiv 1$ and $q = -m^2/r$, and type (i) and (ii) boundary conditions ($R(1) = 0$ and we demand regularity at $r = 0$ with $p(0) = 0$). It is also a form of the general ODE that has Bessel functions as solutions with

$$\alpha = 0, \quad \frac{1}{4}\omega^2 = \lambda, \quad \beta = \frac{1}{2}, \quad \frac{1}{4}\nu^2 = m^2.$$

The solutions are therefore $J_{2m}(2\sqrt{\lambda r})$ and $Y_{2m}(2\sqrt{\lambda r})$. However, the latter cannot satisfy the regularity condition at $r = 0$ The other boundary condition therefore implies that $J_{2m}(2\sqrt{\lambda}) = 0$, which demands that $2\sqrt{\lambda}$ is a zero of $J_{2m}(z)$. Denoting the n^{th} such zero by z_{mn}, we have $R \propto J_{2m}(z_{mn}\sqrt{r})$.

A general solution of the PDE is therefore

$$u(r,\theta,t) = \sum_{n=1}^{\infty} \left\{ \frac{1}{2} A_{0n} J_0(z_{0n}\sqrt{r}) e^{-z_{0n}^2 t/4} + \sum_{m=1}^{\infty} (A_{mn}\cos m\theta + B_{mn}\sin m\theta) J_{2m}(z_{mn}\sqrt{r}) e^{-z_{mn}^2 t/4} \right\}.$$

At $t = 0$, and exploiting a Fourier series for the initial condition, we need

$$u(r,\theta,0) = f(r,\theta) = \frac{1}{2} a_0(r) + \sum_{m=1}^{\infty} [a_m(r)\cos m\theta + b_m(r)\sin m\theta].$$

Given the Sturm-Liouville expansion formulae, we may enforce this by setting

$$A_{0n} = \frac{\int_0^1 a_0(r) J_0(z_{0n}\sqrt{r}) dr}{\int_0^1 [J_0(z_{0n}\sqrt{r})]^2 dr}, \qquad [A_{mn}, B_{mn}] = \frac{\int_0^1 [a_m(r), b_m(r)] J_{2m}(z_{mn}\sqrt{r}) dr}{\int_0^1 [J_{2m}(z_{mn}\sqrt{r})]^2 dr}.$$

(b) When the initial condition has no $\theta-$dependence and $u(r,0) = f(r)$, $A_{mn} = B_{mn} = 0$. The long-time behaviour of the solution is then controlled by the smallest value of $z_{0n}^2/4$ (the exponent of the slowest decaying term in the remaining sum). This is given by the first zero of $J_0(z)$, which is $z \approx 2.40$. The long-time decay rate is therefore $(2.40)^2/4 \approx 1.45$, as observed in the numerical solution.

(c) If $f(r,\theta) = 16r^2(1-r)^2 \sin\theta$, the entire solution has the factor $\sin\theta$ with only the coefficients B_{1n} non-zero. The revised figure shows a comparison of the numerical solution with the analytical one, truncated to five terms. The updated code pde20bx.m performs the task. Note that, since $u(r,\theta,t) \propto \sin\theta$, one can factor out the $\theta-$dependence and plot the solution as a function of only r and t, as done in the figure (or, equivalently, one could take the nominal value of θ of $\pi/2$, for illustration).

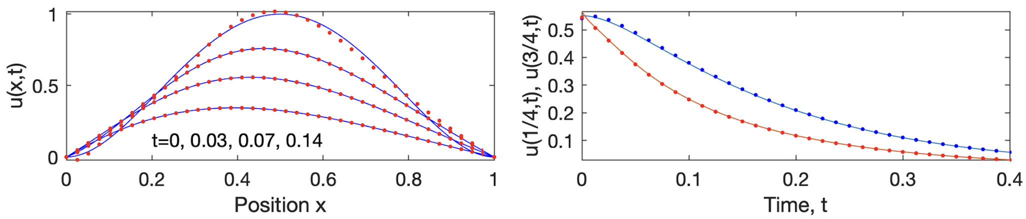

Figure 4: Comparison of numerical and truncated analytical solutions.

Solutions to actual problems

1. (18 points)

The steady state solution is $U = 1 - r^2$. *(2 points).*

After introducing the new variable v, we arrive at the problem

$$v_t = \nabla^2 v, \quad v(1,\theta,t) = 0, \quad v(r,\theta,0) = \cos\theta - (1-r^2) \qquad (1\ \textit{point}).$$

We separate variables: $v = R(r)\Theta(\theta)T(t)$, giving

$$\frac{T'}{T} = \frac{(rR')'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta}.$$

This function or t, or function of r and θ must then equal the separation constant $-\omega^2$, leading to

$$T = Ce^{-\omega^2 t} \quad \& \quad \frac{r(rR')'}{R} + \omega^2 r^2 = -\frac{\Theta''}{\Theta} \qquad (3\ \textit{points}).$$

We therefore introduce a second separation constant m^2 to find

$$\Theta = \textit{constant} \times \begin{cases} 1, & m = 0, \\ \cos m\theta \text{ or } \sin m\theta, & m = 1, 2, ..., \end{cases} \qquad \& \quad R \propto J_m(\omega r).$$

We only need the $m = 0$ term and the $\cos\theta$ solution for $m = 1$ in view of the initial condition. Hence, a suitable general solution is

$$v = \sum_{j=1}^{\infty} \left[c_j J_0(\omega_{0j} r) e^{-\omega_{0j}^2 t} + C_j J_1(\omega_{1j} r) e^{-\omega_{1j}^2 t} \cos\theta \right],$$

where the boundary condition at $r = 1$ tells us that ω_{mj} is fixed as a zero of a Bessel function: $J_m(\omega_{mj}) = 0$. *(4 points).*

To find the constants c_j and C_j we apply the initial condition on v:

$$c_j = -\frac{\int_0^1 (1-r^2) J_0(\omega_{0j} r) r dr}{\int_0^1 [J_0(\omega_{0j} r)]^2 r dr}, \quad C_j = \frac{\int_0^1 J_1(\omega_{1j} r) r dr}{\int_0^1 [J_1(\omega_{1j} r)]^2 r dr} \qquad (2\ \textit{points}).$$

Finally, we reconstruct $u(r,\theta,t) = U(r) + v(r,\theta,t)$ *(1 point).*

The two parts to the solution coincide with the two numerical solutions of the figure. Constructing the truncated series and plotting the results gives the modification to the lowest panels shown below. The truncated series for $m = 0$ is pretty good for all times and radii; the difference between the series and numerical solution is less that 3.3×10^{-3} for the data in the figure. The truncated series solution for $m = 1$ is plagued by Gibbs phenomenon for $t \to 0$ and $r \to 0$ or $r \to 1$. Otherwise, it works well. *(5 points, including the plots).*

2 (10 points) We separate variables: $u = R(r)T(t)$, giving

$$\frac{T'}{T} = \frac{(r^2 R')'}{r^2 R} + \frac{3}{16 r^2}.$$

This function or t or r must then equal a separation constant $-\omega^2$, leading to

$$T = Ce^{-\omega^2 t} \quad \& \quad (r^2 R')' + \omega^2 r^2 R + \frac{3}{16} R = r^2 R'' + 2rR' + \omega^2 r^2 R + \frac{3}{16} R = 0 \qquad (2\ \textit{points}).$$

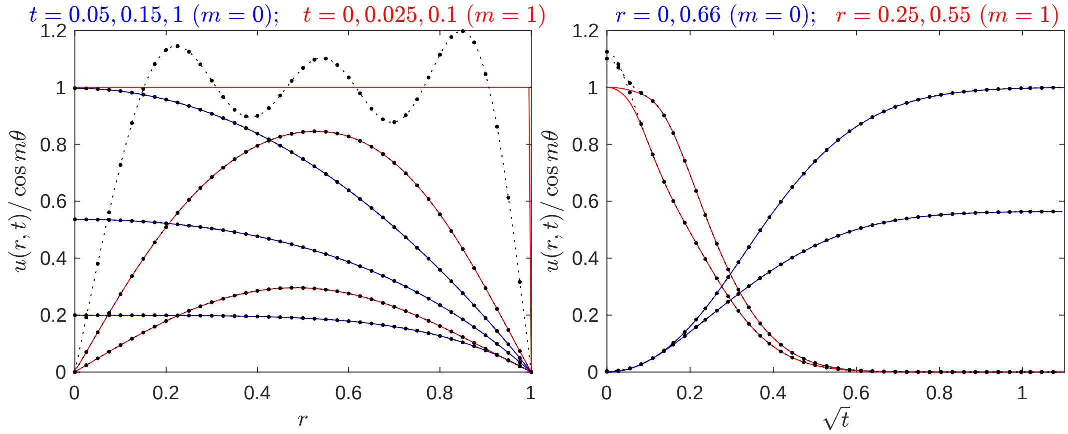

Figure 5: Comparison of the numerical and truncated analytical solutions.

We can match up the remaining ODE with the more general form of Bessel's equation by taking

$$\alpha = -\frac{1}{2}, \quad \beta = 1, \quad \nu = \frac{1}{4},$$

implying the solution

$$R \propto \frac{J_{\frac{1}{4}}(\omega r)}{\sqrt{r}},$$

after discarding the other Bessel function, which fails to respect the regularity condition needed for $r \to 0$ *(3 points)*. Note that the weight function of the ODE is r^2. The boundary condition implies that ω must be taken to be a zero of $J_{\frac{1}{4}}(\omega)$. We denote the j^{th} such zero as ω_j, and arrive at the general solution,

$$u(r,t) = \sum_{j=1}^{\infty} c_j \frac{J_{\frac{1}{4}}(\omega_j r)}{\sqrt{r}} e^{-\omega_j^2 t},$$

where

$$c_j = \frac{\int_0^1 J_{\frac{1}{4}}(\omega_j r) r^{3/2} dr}{\int_0^1 [J_{\frac{1}{4}}(\omega_j r)]^2 r dr} \qquad (3\ \textit{points}).$$

The luminosity is given by the exponentially decaying function,

$$L = -\sum_{j=1}^{\infty} \omega_j c_j J'_{\frac{1}{4}}(\omega_j) e^{-\omega_j^2 t} \qquad (1\ \textit{point}).$$

The first term dominates at large times, implying a half-life of approximately $\ln 2/\omega_1^2 \approx 0.0896$ (in whatever cosmic units are at work here) *(1 point; no need for the numerical value)*.