Coursework 5: Method of characteristics

(1) Using the method of characteristics, solve

$$u_t - (x + \cos t)u_x + u = 0,$$

for $-\infty < x < \infty$, subject to $u(x, 0) = \sin x$.

(2) Using the method of characteristics, solve

$$u_t + 2t(t^2 + x)u_x = 4xt,$$

for $x \geq 0$, subject to $u(x, 0) = F(x)$ and $u(0, t) = f(t)$.

(3) Solve

$$u_t + u^3 u_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(x, 0) = \frac{\sigma x}{\sqrt{1 + x^2}},$$

for (a) $\sigma = +1$ and (b) $\sigma = -1$. In each case, establish whether the solution continues to exist for all time, finding when and where things go wrong if all goes pear-shaped. Sketch the characteristics diagram and sample snapshots of your solution, indicating any breakdown of the solution. For the shocking case, provide a formula for the speed of any discontinuities and qualitatively describe the shock dynamics. Sketch a solution for $t = 40$ in which an equal-area rule is used to avoid a multi-valued solution.

Sample problems

(1) Using the method of characteristics, solve

$$x^2 u_t + u_x = u^{-2}$$

for $-\infty < x < \infty$ and $t > 0$, subject to $u(x,0) = f(x)$.

The characteristic equations are

$$\frac{dx}{dt} = \frac{1}{x^2} \quad \& \quad \frac{du}{dt} = -\frac{1}{x^2 u^2}.$$

Hence, given that $x = x_0$ *and* $u = f(x_0)$ *at* $t = 0$,

$$x^3 = x_0^3 + 3t \quad \& \quad u^3 = [f(x_0)]^3 + 3(x_0^2 + 3t)^{1/3} - 3x_0.$$

Replacing x_0 *by* $(x^3 + 3t)^{1/3}$ *in the latter gives* $u(x,t)$.

(2) Using the method of characteristics, solve

$$u_t + x(1-x)u_x = x,$$

for $-\infty < x < \infty$, subject to $u(x,0) = 0$.

The characteristic equations are

$$\frac{dx}{dt} = x(1-x) \quad \& \quad \frac{du}{dt} = x.$$

Hence, given that $x = x_0$ *and* $u = 0$ *at* $t = 0$,

$$x = \frac{x_0 e^t}{1 - x_0 + x_0 e^t} \quad \text{or} \quad x_0 = \frac{x}{x + (1-x)e^t} \quad \& \quad u = \log[1 + x_0(e^t - 1)] = \log\left[1 + \frac{x(e^t - 1)}{x + (1-x)e^t}\right].$$

(3) Using the method of characteristics, solve

$$x^2 u_t + u_x = x^2$$

for $x > 0$, subject to $u(x,0) = 0$ and $u(0,t) = f(t)$. Provide a condition on $f(t)$ which guarantees that the solution is continuous.

The characteristic equations are

$$\frac{dx}{dt} = x^{-2} \quad \& \quad \frac{du}{dt} = 1.$$

Hence, if $x = x_0$ *and* $u = 0$ *at* $t = 0$,

$$x^3 = x_0^3 + 3t \quad \& \quad u = t$$

which is the case for $x^3 > 3t$. *But if the characteristic leaves* $x = 0$ *at* $t = t_0$ *with* $u = f(t_0)$, *we find instead that*

$$x^3 = 3(t - t_0) \quad \& \quad u = f(t_0) + t - t_0 = f(t - x^3/3) + x^3/3.$$

If $f(0) = 0$, *then* $u \to x^3/3 = t$ *along the dividing characteristic curve* $t = x^3/3$, *rendering the solution continuous.*

Coursework 5: Solutions to actual problems

(1) (2 points) The characteristic equations are

$$\frac{dx}{dt} = -x - \cos t \quad \& \quad \frac{du}{dt} = -u,$$

giving

$$x = x_0 e^{-t} + \frac{1}{2}(e^{-t} - \cos t - \sin t) \quad \& \quad u = e^{-t} \sin x_0.$$

Eliminating x_0 gives

$$u = e^{-t} \sin \left[x e^t - \frac{1}{2}(1 - e^t \cos t - e^t \sin t) \right]$$

(2) (4 points) The characteristic equations are

$$\frac{dx}{dt} = 2tx + 2t^3 \quad \& \quad \frac{du}{dt} = 4xt.$$

Hence, if the characteristic intersects $x = x_0$ and $u = F(x_0)$ at $t = 0$,

$$x = (1 + x_0)e^{t^2} - 1 - t^2 \quad \& \quad u = F(x_0) + 2(1 + x_0)e^{t^2} - 2 - 2t^2 - t^4,$$

giving

$$x_0 = (x + 1 + t^2)e^{-t^2} - 1 \quad \& \quad u(x,t) = F(x_0) + 2x - t^4 \qquad \text{for } x > e^{t^2} - 1 - t^2.$$

But if the characteristic leaves $x = 0$ at $t = t_0$ with $u = f(t_0)$, we find instead, for $x < e^{t^2} - 1 - t^2$, that

$$x = (1 + t_0^2)e^{t^2 - t_0^2} - 1 - t^2 \quad \& \quad u = f(t_0) + t_0^4 - 2 - t^4 - 2t^2 + 2(1 + t_0^2)e^{t^2 - t_0^2},$$

for which the solution is given only implicitly.

(3) (12 points) The characteristic equations are

$$\frac{dx}{dt} = u^3 \quad \& \quad \frac{du}{dt} = 0.$$

Hence, given $u = f(x_0)$ at $t = 0$,

$$x = x_0 + u^3 t \quad \text{and} \quad u = f(x_0) = \frac{\sigma x_0}{\sqrt{1 + x_0^2}}.$$

That is, the implicit solution,

$$u = \frac{\sigma(x - u^3 t)}{\sqrt{1 + (x - u^3 t)^2}},$$

which can be graphed easily at least.

For $\sigma = +1$, the solution spreads out, shallowing with time. With $\sigma = -1$, however, the solution steepens, both above $u = \frac{1}{2}$ and below $u = -\frac{1}{2}$. Shocks form when u_x first diverges:

$$u_x = \frac{f'(x_0)}{1 + 3t[f(x_0)]^2 f'(x_0)}$$

That is, for

$$t = t_s = -\{3[f(x_0)]^2 f'(x_0)\}^{-1}$$

where x_0 gives the minimum of $(f^3)'$. Hence $x_0 = \pm\sqrt{2/3}$ and $t_s \approx 1.793$. These translate to initial shock positions of $x_*(t_s) \approx 0.363$. Afterwards, the solution is multi-valued.

The speed of any shock (at position $x = x_*(t)$) is given by

$$\frac{dx_*}{dt} = \frac{(u^+)^4 - (u^-)^4}{4(u^+ - u^-)}$$

where $u^\pm$ denote the limiting values of u to either side.

Two shocks form at the shock time predicted above. These discontinuities then move toward each other and collide at $x = 0$. Afterwards, since the profile is symmetrical about $u = 0$ and $u^+ = -u^-$, the merged shock stays still.