Complex numbers review problems

1. For which (real) values of $\alpha$ are the solutions to $r^2 + \alpha r + 1 = 0$ real numbers?

Solving the quadratic:

$$r = \frac{1}{2}(-\alpha \pm \sqrt{\alpha^2 - 4})$$

The argument of the square root is positive when $\alpha^2 > 4$ (*i.e.* when $\alpha > 2$ or $\alpha < -2$), implying real roots for $r$. When $-2 < \alpha < 2$ the roots are complex.

2. For which (real) values of $\beta$ are the solutions to $r^2 + i\beta r + 1 = 0$ purely imaginary numbers? And if $r^2 + i\beta r - 1 = 0$?

Solving the quadratic:

$$r = \frac{1}{2}(-i\beta \pm \sqrt{-\beta^2 - 4}) = \frac{i}{2}(\pm\sqrt{\beta^2 + 4} - \beta)$$

Since $\beta^2 + 4 > 0$, the roots are always purely imaginary. For the second case

$$r = \frac{1}{2}(-i\beta \pm \sqrt{4 - \beta^2}).$$

The roots have non-zero real part when $4 > \beta^2$ ($-2 < \beta < 2$), and are purely imaginary otherwise.

3. Write the complex number $(1+i)/(1-i)$ in the forms $x + iy$ and $re^{i\theta}$.

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{(1+i)(1-i)} = \frac{1}{2}(1 + 2i - 1) = i$$

Hence $x = 0$ and $y = 1$. We also have that $r = \sqrt{x^2 + y^2} = 1$ and $\theta = \tan^{-1}(y/x) = \pi/2$.

Or (in a more roundabout and enjoyable way),

$$e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = (1+i)/\sqrt{2} \qquad \& \qquad e^{-i\pi/4} = \cos(\pi/4) - i\sin(\pi/4) = (1-i)/\sqrt{2}$$

Thus,

$$\frac{1+i}{1-i} = \frac{e^{i\pi/4}}{e^{-i\pi/4}} = e^{i\pi/2} = i.$$

We immediately read off $x = 0$ $y = r = 1$ and $\theta = \pi/2$.

4. Rewrite the function $c_1 e^{(-2+3i)t} + c_2 e^{(-2-3i)t}$ in the form $a_1 e^{-\alpha t}\cos(\beta t) + a_2 e^{-\alpha t}\sin(\beta t)$. Show that this also equals $Re^{-\alpha t}\cos(\beta t + \gamma)$ if $R = \sqrt{a_1^2 + a_2^2}$ and $\tan\gamma = -a_2/a_1$.

Euler's formula is $e^{i\theta} = \cos\theta + i\sin\theta$. Hence,

$$c_1 e^{(-2+3i)t} + c_2 e^{(-2-3i)t} = (c_1 e^{+3it} + c_2 e^{-3it})e^{-2t} = [(c_1 + c_2)\cos(3t) + i(c_1 - c_2)\sin(3t)]e^{-2t},$$

and so $a_1 = c_1 + c_2$, $a_2 = i(c_1 - c_2)$, $\alpha = 2$ and $\beta = 3$.

With the definitions of $R$ and $\gamma$, we have $\cos\gamma = a_1/R$ and $\sin\gamma = -a_2/R$. Hence

$$[a_1\cos(\beta t) + a_2\sin(\beta t)]e^{-\alpha t} = \sqrt{a_1^2 + a_2^2}\left[\frac{a_1\cos(\beta t)}{\sqrt{a_1^2 + a_2^2}} + \frac{a_2\sin(\beta t)}{\sqrt{a_1^2 + a_2^2}}\right]e^{-\alpha t} = R(\cos(\beta t)\cos\gamma - \sin(\beta t)\sin\gamma)e^{-\alpha t}$$

which gives the final result on using a trig formula.