Part I.

1) Evalues $\quad (\lambda+2)^2 - 4 = 0 \rightarrow \lambda = 0, -4$ (d)

2) $\underset{\sim}{y}_p = (p^2 I - A)^{-1} \underset{\sim}{f}_0 e^{pt}$ $\quad$ provided p^2 not an eigenvalue

(d)

3) $\dfrac{4}{s^2+4} + \dfrac{1}{s-3} \rightarrow$ (c)

4) $(s^2+3s+2)\bar{y} = \dfrac{3}{s^2} \rightarrow \bar{y} = \dfrac{3}{s^2(s+1)(s+2)}$

(d)

5) $\dfrac{1}{s-3} - \dfrac{1}{s-1} \longrightarrow e^{3t} - e^{t}$ $\quad$ (b)

Part II.

1) Rearranging the ODEs... $\quad x'' = -x - 2y + e^{2t}$

$\quad y' = -2x - y$

12 pts

$$\therefore \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$$

Hom. Sols. $\quad$ Put $\begin{pmatrix} x \\ y \end{pmatrix} = \underline{v} e^{mt} \implies m^2 \underline{V} = A \underline{V}$

$\therefore$ m^2 is an eigenvalue λ of A & $\underline{v}$ the eigenvector corresponding to it.

3pts

$$\det(A - \lambda I) = 0 \implies (1+\lambda)^2 - 4 = 0$$

$$\lambda^2 + 2\lambda - 3 = 0 \qquad (\lambda+3)(\lambda-1) = 0$$

$$\lambda = 1: \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \underline{v} = \underline{0} \implies \underline{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\underline{\lambda = -3} \rightarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \underline{v} = \underline{0} \rightarrow \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 4pts

Part. Sol. try $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{2t} \Rightarrow$ 1pt

$$4 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

ie. $5\alpha + 2\beta = 1$ & $5\beta + 2\alpha = 0$

$\rightarrow \alpha = \frac{5}{21}$ & $\beta = \frac{-2}{21}$ 2pts

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \overbrace{\left(A_1 e^{t} + A_2 e^{-t}\right)}^{(m^2=1)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \overbrace{\left(B_1 \cos\sqrt{3}t + B_2 \sin\sqrt{3}t\right)}^{(m^2=-3)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 5/21 \\ -2/21 \end{pmatrix} e^{2t}$$ 2pts

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2). 9 pts $\mathcal{L}\{e^{at} y(t)\} = \int_0^\infty e^{-st+at} y(t)\, dt$

$= \underline{\bar{y}(s-a)}$ since $\bar{y}(s) = \int_0^\infty e^{-st} y(t)\, dt$

$\mathcal{L}\{t^n\} = n!/s^{n+1}$

$\therefore \mathcal{L}\{e^{at} t^n\} = n!/(s-a)^{n+1}$ (3pts)

LT the ODE... (2pts)

$$s^2 \bar{y} - s + 2(s\bar{y} - 1) + \bar{y} = \mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}$$

result above with $a=-1$

Hence $(s+1)^2\bar{y} = s+2+\dfrac{1}{(s+1)^2}$

(2pts)

$$\therefore \bar{y} = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)^4}$$

$$\therefore y = e^{-t} + te^{-t} + \frac{1}{3!}t^3e^{-t}$$

(2pts)

using the previous result again