I.

1. $y' + xy = xe^{x^2/2}$, $y(0)=1$

$I(x) = e^{x^2/2} \longrightarrow y = \left(C + \int xe^{x^2}dx\right)\frac{1}{I}$

$= Ce^{-x^2/2} + \frac{1}{2}e^{x^2/2}$

$\therefore\ y = \frac{1}{2}\left(e^{x^2/2} + e^{-x^2/2}\right)$ (a)

2. $y' = y^2 f(x) \longrightarrow \int \frac{dy}{y^2} = \int f(x)dx + \hat{C}$

$-\frac{1}{y} = \int f(x)dx + \hat{C}$, $\hat{C} = -C$

$y = \left[C - \int f(x)\,dx\right]^{-1}$ (b)

3. $y'' + 6y' + 25y = 0 \rightarrow m^2 + 6m + 25$

$(m+3)^2 + 16 = 0 \rightarrow m = -3 \pm 4i$

(b)

4. Try $y_p = ax^2 + bx + c \rightarrow 2a\,\cancel{4}\,3(2ax+b) - 4(ax^2+bx+c) = -4x^2$

$\therefore -4a = -4$ or $a = 1$

$-6ax - 4bx = 0$ or $b = -3/2$

& $2a - 3b = 4c$ or $c = 13/8$ (a)

II

1. $I = \exp\left(\int \ln x\,dx\right) = \exp\left(x\ln x - \int x\frac{1}{x}dx\right) = x^x e^{-x}$

$\int qI\,dx = \int x^{-x}x^x e^{-x}dx = -e^{-x}$

$\therefore\ y = \left(C - e^{-x}\right)e^x x^{-x}$

$y(1) = 0 \Rightarrow 0 = C - e^{-1}$, $C = e^{-1}$

$\Rightarrow y = e^{x-1}x^{-x} - x^{-x}$ (so $A = 1$)

2. $\cancel{2}y'' + \cancel{2}y' - \overset{2}{\cancel{4}}y = \overset{5}{\cancel{10}}\cos x + \overset{5}{\cancel{10}}\sin x$

Hom. Sols. $m^2 + m - 2 = 0 \rightarrow (m+2)(m-1) = 0$

$y_h = A_1 e^{-2x} + A_2 e^{x}$.

Part Sols. $y_p = d_1 \cos x + d_2 \sin x$

$\rightarrow (-d_1 + d_2 - 2d_1)\cos x + (-d_2 - d_1 - 2d_2)\sin x$

$= 5\cos x + 5\sin x$

$d_2 - 3d_1 = 5 \rightarrow d_2 = 3d_1 + 5$

$-3d_2 - d_1 = 5 \longrightarrow -9d_1 - 15 - d_1 = 5$

$\therefore d_1 = -2$

$\&\ d_2 = -1$

So $y = A_1 e^{-2x} + A_2 e^{x} - 2\cos x - \sin x$

$y(0) = y'(0) = 0 \Rightarrow A_1 + A_2 = 2$ $\&\ -2A_1 + A_2 = 1$ $\Rightarrow 3A_1 = 1$ or $A_1 = \frac{1}{3}$, $A_2 = \frac{5}{3}$

$$y = \frac{1}{3}e^{-2x} + \frac{5}{3}e^{x} - 2\cos x - \sin x$$

I.

1. $y' - yp(x) = 0 \longrightarrow I = e^{-\int p\,dx} = J$ & $y = \frac{c}{I}$ (d)

2. $y' + \frac{f(x)}{y} = 0 \rightarrow \frac{1}{2}y^2 + \int f(x)\,dx = \hat{C}$ $\quad \hat{C} = \frac{1}{2}C$

$$y = \pm\sqrt{C - 2\int f(x)\,dx}$$ (b)

3. $y'' - 4y' + 5y = 0 \rightarrow m^2 - 4m + 5 = 0$

$(m-2)^2 + 1 = 0 \qquad m = 2 \pm i$ (a)

4. $y'' + y' + 2y = 4x^2$, try $y_p = ax^2 + bx + c$

$$2a + 2ax + b + 2(ax^2 + bx + c) = 4x^2$$

$\rightarrow a = 2, \quad 2ax + 2bx = 0 \rightarrow b = -2$

$2a + b + 2c = 0 \longrightarrow c = -1$

$$y_p = 2x^2 - 2x - 1$$ (b)

II. 1. $\cancel{(1-x^2)}\,y' - \frac{xy}{(1-x^2)} = \frac{\sqrt{1-x^2}\,(1+x^2)^2}{(1-x^2)}$

$$I = \exp\left(-\int \frac{x\,dx}{1-x^2}\right) = \exp\Big(-\frac{1}{2}\underbrace{\int \frac{du}{1-u}}_{u=x^2}\Big) = \sqrt{1-x^2}$$

$$\therefore \left(\sqrt{1-x^2}\,y\right)' = (1+x^2)^2 = 1 + 2x^2 + x^4$$

$$y = \cancel{\frac{C}{\sqrt{1-x^2}}} + \frac{\left(x + \frac{2}{3}x^3 + \frac{1}{5}x^5\right)}{\sqrt{1-x^2}}$$

$y(0) = 0 \Rightarrow C = 0$.

2. $y'' - 4y' + 4y = e^{\lambda x}$. Hom. Sols: $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$$\rightarrow y_h = (Ax + B)e^{2x}$$

If $\lambda \neq 2$, RHS is not a homog. sol.

$\therefore$ try $y_p = ae^{\lambda x}$

$$\Rightarrow (\lambda-2)^2 a = 1$$

So $$y = (Ax+B)e^{2x} + \frac{e^{\lambda x}}{(\lambda-2)^2}$$

$y(0) = y'(0) = 0 \Rightarrow$

$$B = -\frac{1}{(\lambda-2)^2}$$

$$\& \quad A + 2B + \frac{\lambda}{(\lambda-2)^2} = 0$$

$$\text{so } A = -\frac{1}{(\lambda-2)}$$

$$\therefore y = \frac{e^{\lambda x} - e^{2x} - (\lambda-2)xe^{2x}}{(\lambda-2)^2}$$

If $\lambda = 2$, the RHS is a hom. Sol. $\Rightarrow$ neither e^{2x} nor xe^{2x} will work!

Try $y_p = ax^2e^{2x} \Rightarrow$

$$(\cancel{4ax^2} + \cancel{8ax} + 2a)e^{2x} - 4(\cancel{2ax^2} + \cancel{2ax})e^{2x} + \cancel{4ax^2e^{2x}} = e^{2x}$$

$\therefore a = \frac{1}{2}$. $\quad y = (Ax+B)e^{2x} + \frac{1}{2}x^2e^{2x}$

$y(0) = 0 \Rightarrow B = 0$, $y'(0) = 0 \Rightarrow A = 0$

$$\therefore y = \frac{1}{2}x^2e^{2x}$$