Math 256. Midterm 2.

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Name:

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The system

$$\mathbf{y}' = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{y}$$

has the general solution,

$$(a) \quad \mathbf{u}_1 e^{t} + \mathbf{u}_2 e^{-t} \qquad (b) \quad \mathbf{u}_1 + \mathbf{u}_2 e^{2t} \qquad (c) \quad \mathbf{u}_1 t + \mathbf{u}_2 e^{4t}$$

$$(d) \quad \mathbf{u}_1 + \mathbf{u}_2 e^{-4t} \qquad (e) \quad \textit{None of the above,}$$

for two constant vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

2. If λ_1 and λ_2 are the eigenvalues of the constant matrix A, the particular solution to

$$\mathbf{y}'' = A\mathbf{y} + \mathbf{f}_0 e^{pt},$$

where p is a constant and $\mathbf{f}_0$ a constant vector, is given by

$$(a) \quad (p^2 I - A)^{-1}\mathbf{f}_0 e^{pt} \ \ \textit{for any } p \qquad (b) \quad (pI - A)^{-1}\mathbf{f}_0 e^{pt} \ \ \textit{for } p \neq \lambda_1 \textit{ or } \lambda_2$$

$$(c) \quad (pI - A)^{-1}\mathbf{f}_0 e^{pt} \ \ \textit{for any } p \qquad (d) \quad (p^2 I - A)^{-1}\mathbf{f}_0 e^{pt} \ \ \textit{for not all } p$$

$$(e) \quad \textit{None of the above}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are constant vectors.

3. The Laplace transform of $2\sin 2t + e^{3t}$ is

$$(a) \quad \frac{2s}{s^2-4} + \frac{1}{s-3} \qquad (b) \quad \frac{2s}{s^2+4} + \frac{1}{s+3} \qquad (c) \quad \frac{4}{s^2+4} + \frac{1}{s-3}$$

$$(d) \quad \frac{4}{s^2+4} + \frac{1}{s+3} \qquad (e) \quad \textit{None of the above.}$$

4. The Laplace transform of the ODE

$$y'' + 3y' + 2y = 3t, \quad y(0) = y'(0) = 0,$$

yields the $\bar{y}(s)$ given by

$$(a) \quad \frac{3}{s(s-2)(s-1)} \qquad (b) \quad \frac{3}{s^2(s-2)(s-1)} \qquad (c) \quad \frac{3}{s(s+2)(s+1)} \qquad (d) \quad \frac{3}{s^2(s+2)(s+1)}$$

$$(e) \quad \textit{None of the above.}$$

5. The inverse Laplace transform of $2/(s-3)(s-1)$ is

$$(a) \quad e^{-t} - e^{-3t} \qquad (b) \quad e^{3t} - e^{t} \qquad (c) \quad e^{t} + e^{3t}$$

$$(d) \quad e^{-t} + e^{-3t} \qquad (e) \quad \textit{None of the above.}$$

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(12 points)* Two coupled electrical circuits have currents $x(t)$ and $y(t)$ that satisfy the ODEs,

$$x'' + x + 2y = e^{2t}, \quad y'' + y + 2x = 0.$$

Write these equations as a system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system.

2. *(9 points)* Using the definition of the Laplace transform, establish the first shifting theorem $\mathcal{L}\{e^{at}y(t)\} = \overline{y}(s-a)$, where a is a constant. Hence compute the Laplace transform of $t^n e^{at}$. Using Laplace transforms, solve the ODE

$$y'' + 2y' + y = te^{-t}, \qquad y(0) = 1, \quad y'(0) = 0.$$

Useful Laplace Transforms

$f(t)$		$\bar{f}(s)$
1	$\rightarrow$	$1/s$
$t^n, \quad n = 0, 1, 2, ...$	$\rightarrow$	$n!/s^{n+1}$
e^{at}	$\rightarrow$	$1/(s-a)$
$\sin at$	$\rightarrow$	$a/(s^2+a^2)$
$\cos at$	$\rightarrow$	$s/(s^2+a^2)$
$t \sin at$	$\rightarrow$	$2as/(s^2+a^2)^2$
$t \cos at$	$\rightarrow$	$(s^2-a^2)/(s^2+a^2)^2$
$y'(t)$	$\rightarrow$	$s\overline{y}(s) - y(0)$
$y''(t)$	$\rightarrow$	$s^2\overline{y}(s) - y'(0) - sy(0)$
$e^{at}f(t)$	$\rightarrow$	$\bar{f}(s-a)$
$f(t-a)H(t-a)$	$\rightarrow$	$e^{-as}\bar{f}(s)$

Helpful trig identities:

$$\sin 0 = \sin \pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),$$
$$\cos 0 = -\cos \pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$
$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$
$$\sin(2A) = 2\sin A \cos A, \quad \sin(A+B) = \sin A \cos B + \cos A \sin B,$$
$$\cos(2A) = \cos^2 A - \sin^2 A, \quad \cos(A+B) = \cos A \cos B - \sin A \sin B,$$