Math 256. Midterm 2.

No formula sheet, books or calculators!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The system

$$\mathbf{y}' = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \mathbf{y}$$

has the general solution,

$(a)\quad \mathbf{u}_1 e^{t} + \mathbf{u}_2 e^{-t}$ $\quad (b)\quad \mathbf{u}_1 + \mathbf{u}_2 e^{2t}$ $\quad (c)\quad \mathbf{u}_1 t + \mathbf{u}_2 e^{4t}$

$(d)\quad \mathbf{u}_1 + \mathbf{u}_2 e^{4t}$ $\quad (e)$ *None of the above,*

for two constant vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

2. If λ_1 and λ_2 are the eigenvalues of the constant matrix A, the solution to

$$\mathbf{y}' + A\mathbf{y} = \mathbf{f}_0 e^{pt},$$

where p is a constant and $\mathbf{f}_0$ a constant vector, is given by

$(a)\quad \mathbf{u}_1 e^{\lambda_1 t} + \mathbf{u}_2 e^{\lambda_2 t} + (pI - A)^{-1}\mathbf{f}_0 e^{pt}$ *for any* p $\quad (b)\quad \mathbf{u}_1 e^{\lambda_1 t} + \mathbf{u}_2 e^{\lambda_2 t} + (pI - A)^{-1}\mathbf{f}_0 e^{pt}$ *for* $p \neq \lambda_1$ *or* λ_2

$(c)\quad (pI + A)^{-1}\mathbf{f}_0 e^{pt}$ *for any* p $\quad (d)\quad (pI + A)^{-1}\mathbf{f}_0 e^{pt}$ *for certain initial conditions and not all* p

(e) *None of the above*

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are constant vectors.

3. The Laplace transform of $2\cos 5t + e^{6t}$ is

$(a)\quad \dfrac{2s}{s^2 - 25} + \dfrac{1}{s - 6}$ $\quad (b)\quad \dfrac{2s}{s^2 - 25} + \dfrac{1}{s + 6}$ $\quad (c)\quad \dfrac{2s}{s^2 + 25} + \dfrac{1}{s - 6}$

$(d)\quad \dfrac{2s}{s^2 + 25} + \dfrac{1}{s + 6}$ $\quad (e)$ *None of the above.*

4. The Laplace transform of the ODE

$$y'' + 7y' + 6y = 30, \quad y(0) = y'(0) = 0,$$

yields the $\bar{y}(s)$ given by

$(a)\quad \dfrac{30}{(s-6)(s-1)}$ $\quad (b)\quad \dfrac{30}{s(s-6)(s-1)}$ $\quad (c)\quad \dfrac{30}{(s+6)(s+1)}$ $\quad (d)\quad \dfrac{30}{s(s+6)(s+1)}$

(e) *None of the above.*

5. The inverse Laplace transform of $2s/(s^2 - 4)$ is

$(a)\quad e^{2t} + e^{-2t}$ $\quad (b)\quad e^{2t} - e^{-2t}$ $\quad (c)\quad \dfrac{1}{3}e^{2t} + \dfrac{1}{3}e^{-2t}$

$(d)\quad \dfrac{1}{3}e^{2t} - \dfrac{1}{3}e^{-2t}$ $\quad (e)$ *None of the above.*

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. Two masses attached to a string at points $x(t)$ and $y(t)$ have the equations of motion,

$$x'' + \omega^2(2x - y) = 0, \quad y'' + \omega^2(3y - 2x) = 0.$$

Write these equations as a system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. Find the solution if the initial values are $x(0) = x'(0) = y(0) = 0$ and $y'(0) = 1$.

2. Without using the table, show that $\mathcal{L}\{e^{at} - e^{bt}\} = (a-b)/[(s-a)(s-b)]$, for constants a and b, commenting on the range of s for which the transform exists. Invert the Laplace transforms,

$$\frac{2}{(s-1)(s+4)} \quad \text{and} \quad \frac{s+2}{(s-1)(s+4)}$$

Using Laplace transforms, solve the ODEs

$$x' + x - 3y = 0, \qquad y' - 2x + 2y = 0, \qquad x(0) = 1, \quad y(0) = 0.$$

Useful Laplace Transforms

$f(t)$	$\to$	$\bar{f}(s)$
1	$\to$	$1/s$
$t^n, \quad n = 0, 1, 2, ...$	$\to$	$n!/s^{n+1}$
e^{at}	$\to$	$1/(s-a)$
$\sin at$	$\to$	$a/(s^2 + a^2)$
$\cos at$	$\to$	$s/(s^2 + a^2)$
$t \sin at$	$\to$	$2as/(s^2 + a^2)^2$
$t \cos at$	$\to$	$(s^2 - a^2)/(s^2 + a^2)^2$
$y'(t)$	$\to$	$s\bar{y}(s) - y(0)$
$y''(t)$	$\to$	$s^2\bar{y}(s) - y'(0) - sy(0)$
$e^{at}f(t)$	$\to$	$\bar{f}(s-a)$
$f(t-a)H(t-a)$	$\to$	$e^{-as}\bar{f}(s)$