Math 256. Sample midterm 2.

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The system

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{y}$$

has the general solution,

(a) $\mathbf{u}_1 e^t + \mathbf{u}_2 e^{-t}$ (b) $\mathbf{u}_1 e^{5t} + \mathbf{u}_2 e^{-t}$ (c) $\mathbf{u}_1 e^t + \mathbf{u}_2 e^{-3t}$

(d) $\mathbf{u}_1 e^t + \mathbf{u}_2 e^{3t}$ (e) *None of the above,*

for two constant vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

2. A particular solution to

$$\mathbf{y}'' = A\mathbf{y} + \mathbf{f}_0 \cos pt,$$

where A is a constant matrix and f_0 a constant vector, is given by $-(p^2 I + A)^{-1} \mathbf{f}_0 \cos pt$, provided that

(a) $-p^2$ *not an eigenvalue of* A (b) $-p^2$ *not an eigenvalue of* A *and for certain initial conditions*

(c) p *not an eigenvalue of* A (d) p *not an eigenvalue of* A *and for certain initial conditions*

(e) *None of the above.*

3. The Laplace transform of $e^{-at} \sin t$ is

(a) $\dfrac{1}{s^2+1} + a$ (b) $\dfrac{a}{s^2+1}$ (c) $\dfrac{1}{(s+a)^2+1}$ (d) $\dfrac{1}{(s-a)^2+1}$ (e) *None of the above,*

with a a constant.

4. The Laplace transform of the ODE

$$y'' - 3y' + 2y = 0 \quad y(0) = 3, \quad y'(0) = 4,$$

yields the $\bar{y}(s)$ given by

(a) $\dfrac{3s-5}{(s-2)(s-1)}$ (b) $\dfrac{3s+5}{(s-2)(s-1)}$ (c) $\dfrac{3s-5}{(s+2)(s+1)}$ (d) $\dfrac{3s+5}{(s+2)(s+1)}$

(e) *None of the above.*

Extra The answer to the last question is

(a) $e^{2t} + 2e^t$ (b) $e^{2t} + 3e^t$ (c) $e^{-2t} + 2e^{-t}$ (d) $e^{-2t} + 3e^{-t}$ (e) *None of the above.*

5. The inverse Laplace transform of $(s^2 + 6s + 25)^{-1}$ is

(a) $\dfrac{1}{4} e^{3t} \sin 4t$ (b) $\dfrac{1}{3} e^{3t} \sin 4t$ (c) $\dfrac{1}{4} e^{-3t} \sin 4t$ (d) $\dfrac{1}{3} e^{-3t} \sin 4t$ (e) *None of the above.*

5∗. The inverse Laplace transform of $(2s-1)/[(s+2)(s-3)]$ is

(a) $e^{-2t} + e^{3t}$ (b) $\dfrac{1}{3} e^{-2t} + \dfrac{1}{4} e^{3t}$ (c) $e^{2t} + e^{-3t}$ (d) $\dfrac{1}{3} e^{2t} + \dfrac{1}{4} e^{-3t}$ (e) *None of the above.*

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. Two coupled pendulums are described by the angles θ and ϕ, and evolve according to the ODEs,

$$\theta'' = \phi - 3\theta, \quad \phi'' = 2(\theta - \phi).$$

Write these equations as a system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system.

2. Prove that $\mathcal{L}\{y'\} = s\bar{y}(s) - y(0)$ and $\mathcal{L}\{y''\} = s^2\bar{y}(s) - sy(0) - y'(0)$. Using Laplace transforms, solve the ODE

$$y'' - 5y' + 6y = 2e^t, \quad y(0) = y'(0) = 0.$$

Useful Laplace Transforms

$f(t)$	$\to$	$\bar{f}(s)$
1	$\to$	$1/s$
$t^n, \quad n = 0, 1, 2, ...$	$\to$	$n!/s^{n+1}$
e^{at}	$\to$	$1/(s-a)$
$\sin at$	$\to$	$a/(s^2+a^2)$
$\cos at$	$\to$	$s/(s^2+a^2)$
$t \sin at$	$\to$	$2as/(s^2+a^2)^2$
$t \cos at$	$\to$	$(s^2-a^2)/(s^2+a^2)^2$
$y'(t)$	$\to$	$s\bar{y}(s) - y(0)$
$y''(t)$	$\to$	$s^2\bar{y}(s) - y'(0) - sy(0)$
$e^{at}f(t)$	$\to$	$\bar{f}(s-a)$
$f(t-a)H(t-a)$	$\to$	$e^{-as}\bar{f}(s)$