Part I

1. $\det(A-\lambda I) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) = 0$ for $\lambda = 1, 3$

$\longrightarrow$ (d)

2. part sol. : $\underset{\sim}{y}_p = \underset{\sim}{a}\cos pt \rightarrow -(p^2 I + A)\underline{a} = \underline{f}_0$

$(p^2 I + A)^{-1}$ exists unless $\lambda = -p^2$, one of the eigenvalues. $\longrightarrow$ (a)

(init. cond. irrelevant)

3. $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \therefore \mathcal{L}\{e^{at}\sin t\} = \frac{1}{(s-a)^2+1} \rightarrow$ (d)

4. $\mathcal{L}T$ of ODE $\longrightarrow \quad s^2\bar{y} - 3s - 4 - 3(s\bar{y} - 3) + 2\bar{y} = 0$

$$\longrightarrow \quad \bar{y} = \frac{3s-5}{(s-1)(s-2)} \longrightarrow \text{(a)}$$

Extra: $\bar{y} = \frac{2}{s-1} + \frac{1}{s-2} \longrightarrow y = 2e^t + e^{2t}$

$\rightarrow$ (a)

5. $(s^2+6s+25)^{-1} = [(s+3)^2+16]^{-1} = \frac{1}{4}\,\frac{4}{(s+3)^2+16}$

$= \mathcal{L}\{\frac{1}{4}e^{-3t}\sin 4t\} \rightarrow$ (c)

5* $\frac{2s-1}{(s+2)(s-3)} = \frac{1}{s+2} + \frac{1}{s-3}$

$\rightarrow \mathcal{L}^{-1} = e^{-2t} + e^{3t} \longrightarrow$ (a)

Part II

1. $\begin{pmatrix} \theta'' \\ \phi'' \end{pmatrix} = \underbrace{\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}}_{A} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$. eigenvalues of A: $(\lambda+3)(\lambda+2) - 2$

$= \lambda^2 + 5\lambda + 4 = 0$

$\rightarrow \lambda = -1$ or -4

eigenvectors $\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\underline{v} = \underline{0}$ for $\lambda = -1 \rightarrow \underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

& $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \underline{v} = \underline{0}$ for $\lambda = -4$ $\rightarrow$ $\underline{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore \begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} R_1 \cos(t + \theta_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} R_2 \cos(2t + \theta_2)$

(posing $\underline{y} = \underline{v} e^{mt} \rightarrow m^2 \underline{v} = A\underline{v} \Rightarrow m^2 = \lambda = -1$ or -4
& using the real notation for the general solution)

2. $\mathcal{L}\{y'\} = s\bar{y} - y(0)$ using def & integration by parts

$\mathcal{L}\{y''\} = s^2\bar{y} - sy(0) - y'(0)$ "

LT of ODE : $(s^2 - 5s + 6)\bar{y} = \frac{2}{s-1}$

$\therefore \bar{y} = \frac{2}{(s-1)(s-2)(s-3)} = \frac{a}{s-1} + \frac{b}{s-2} + \frac{c}{s-3}$

Recombining: $a(s^2 - 5s + 6) + b(s^2 - 4s + 3) + c(s^2 - 3s + 2) = 2$

s^2 : $a + b + c = 0$
s : $-5a - 4b - 3c = 0$ $\rightarrow$ $2a + b = 0 \rightarrow b = -2a$
$\rightarrow c = a$
const. : $6a + 3b + 2c = 2$
$\hookrightarrow$ $a = c = 1,\ b = -2$

$\therefore y(t) = \underline{\underline{e^t - 2e^{2t} + e^{3t}}}$

Part I

1. $(2-\lambda)^2 - 4 = \lambda^2 - 4\lambda = 0 \rightarrow \lambda = 0, 4 \rightarrow$ (d)

2. part sol. $\underline{y}_p = \underline{a}e^{pt} \longrightarrow (pI+A)\underline{a} = \underline{f}_0$

$(pI+A)^{-1}$ exists provided $p \neq -\lambda$

$\Rightarrow$ (d) is solution. (init conds can eliminate the homog. sols.)

3. $\mathcal{L}\{2\cos 5t + e^{6t}\} = \dfrac{2s}{s^2+25} + \dfrac{1}{s-6} \longrightarrow$ (c)

4. $\mathcal{L}\{ODE\} \rightarrow (s^2+7s+6)\bar{y} = \dfrac{30}{s}$

$(s+6)(s+1)$ $\longrightarrow$ (d)

5. $\dfrac{2s}{s^2-4} = \dfrac{1}{s+2} + \dfrac{1}{s-2}$, $\mathcal{L}^{-1} \rightarrow e^{2t} + e^{-2t}$ $\longrightarrow$ (a)

Part II

1. ODEs $\rightarrow \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -2\omega^2 & \omega^2 \\ 2\omega^2 & -3\omega^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues: $(\lambda+2\omega^2)(\lambda+3\omega^2) - 2\omega^4 = 0$

$$\lambda^2 + 5\omega^2\lambda + 4\omega^2 = 0$$
$$\lambda = -\omega^2 \text{ or } -4\omega^2$$

eigenvectors:

$\lambda = -\omega^2$, $\begin{pmatrix} -\omega^2 & \omega^2 \\ 2\omega^2 & -2\omega^2 \end{pmatrix} \underline{v} = 0 \rightarrow \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$m^2 = -\omega^2$

$m = \pm i\omega$

$\lambda = -4\omega^2$, $\begin{pmatrix} 2\omega^2 & \omega^2 \\ 2\omega^2 & \omega^2 \end{pmatrix} \underline{v} = 0 \rightarrow \underline{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Posing $\underline{x} = \underline{v}e^{mt} \Rightarrow m^2 = \lambda$. Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = (C_1\cos\omega t + D_1\sin\omega t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (C_2\cos 2\omega t + D_2\sin 2\omega t)\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

At $t=0$, $\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} C_1 + \begin{pmatrix}1\\-2\end{pmatrix} C_2$

& $\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \omega D_1 + \begin{pmatrix}1\\-2\end{pmatrix} 2\omega D_2$

$\rightarrow \quad C_1 = C_2 = 0, \quad 2D_2 = -D_1$

& $D_1 = \dfrac{1}{3\omega}$

$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \dfrac{1}{3\omega}\sin\omega t \begin{pmatrix}1\\1\end{pmatrix} - \dfrac{1}{6\omega}\sin 2\omega t \begin{pmatrix}1\\-2\end{pmatrix}$

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2. Use def & direct integration for proof.

Real(s) > max of a & b for transform to exist

$\mathcal{L}^{-1}\left\{\dfrac{2}{(s-1)(s+4)}\right\} = \dfrac{1}{5}\left(e^{t} - e^{-4t}\right)$ using 1^{st} part.

$\mathcal{L}^{-1}\left\{\dfrac{s+2}{(s-1)(s+4)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s-1} - \dfrac{2}{(s-1)(s+4)}\right\}$

$= e^{t} - \dfrac{2}{5}\left(e^{t} - e^{-4t}\right)$

(or via a partial fraction)

Apply LT to ODEs $\quad (s+1)\bar{x} - 1 - 3\bar{y} = 0$

& $(s+2)\bar{y} = 2\bar{x}$

Eliminate $\bar{y}$ ...

$\left[(s+1)(s+2) - 6\right]\bar{x} = s+2 \Rightarrow x(t)$ is second answer of second part

Similarly, $y(t)$ is second part's first answer.