Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE $y' + y = xe^{-x}$, has the solution,

(a) $C + \frac{1}{2}x^2$ (b) $C + \frac{1}{2}x^2 e^x$ (c) $\left(C + \frac{1}{2}x^2\right) e^{-x}$ (d) $C + \frac{1}{2}x^2 e^{-x}$

(e) *None of the above,*

where C is a constant.

2. The ODE $y' + e^{-y} f(x) = 0$ with $y(0) = 0$, has the solution,

(a) $\ln\left[1 + \int_0^x f(\hat{x}) d\hat{x}\right]$ (b) $\ln\left[1 - \int_0^x f(\hat{x}) d\hat{x}\right]$ (c) $\int_0^x f(\hat{x}) d\hat{x}$

(d) $1 - \exp\left[\int_0^x f(\hat{x}) d\hat{x}\right]$ (e) *None of the above,*

where C is a constant.

3. The ODE $y'' - 4y' + 13y = 0$, has the solution,

(a) $e^{2x}(A\cos 2x + B\sin 2x)$ (b) $e^{-2x}(A\cos x + B\sin x)$ (c) $Ae^{2x}\cos(3x + B)$

(d) $Ae^{(2+3i)x} + Be^{(-2+3i)x}$ (e) *None of the above,*

where A and B are constants.

4. The ODE $y'' + 2y' + y = 2x^2$, has the general solution,

(a) $2x^2 - 8x + 12 + Ae^x + Be^{-x}$ (b) $2x^2 - 8x - 12 + Axe^{-x} + Be^{-x}$ (c) $2x^2 - 8x + 12 + Ae^x + Be^{2x}$

(d) $2x^2 + 8x - 12 + Ae^x + Be^{2x}$ (e) *None of the above.*

12

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. Define the integrating factor for the first-order ODE, $y' + yp(x) = q(x)$.
Hence solve

$$y' \cos x + \left(\frac{\cos x}{x} - \sin x\right) y = \frac{1}{x} e^x \qquad \text{with} \qquad y(0) = 1.$$

Integrating factor: $I(x) = \exp \int p(x)\,dx$ 1

For the ODE, $I = \exp\left[\int \left(\frac{1}{x} - \tan x\right) dx\right]$

$= x \cos x$ 2

$p = \frac{1}{x} - \tan x$

$q = \frac{e^x}{x \cos x}$

Also, $\int qI\,dx = e^x$ 2

$\therefore\ y = \dfrac{C + e^x}{x \cos x}$, where C is arbitrary,

is the general sol. 1

But $y(0) = 1$, so $y = \dfrac{e^x - 1}{x \cos x}$

(with limit $y \to 1$ as $x \to 0$) 2

8 in total

2. Solve the ODE,

$$y'' + 2y' + 2y = x\cos x.$$

Homog. Sols. : aux eq is $m^2 + 2m + 2 = 0$

$$\rightarrow (m+1)^2 + 1 = 0$$

$$m = -1 \pm i$$

$\therefore$ the homog. sols. are $Ae^{-x}\cos x + Be^{-x}\sin x$.

3

Partic. Sol.: we have $x\cos x$ as the inhom. term.

$\therefore$ try a part. sol. $y_p = (ax+b)\cos x + (cx+d)\sin x$

2

Plugging in to ODE :

$$2(ax+b)\cos x + 2(cx+d)\sin x$$
$$+ 2\left[a\cos x + c\sin x - (ax+b)\sin x + (cx+d)\cos x\right]$$
$$+ (-2a)\sin x + 2c\cos x - (ax+b)\cos x - (cx+d)\sin x$$
$$= x\cos x.$$

Match terms:

$\cos x$: $2b + 2a + 2d + 2c - b = 0 \rightarrow b + 2a + 2c + 2d = 0$

$\sin x$: $2d + 2c - 2b - 2a - d = 0 \rightarrow d - 2a + 2c - 2b = 0$

$x\cos x$: $2a + 2c - a = 1 \longrightarrow a + 2c = 1 \longrightarrow a = \frac{1}{5}$

$x\sin x$: $2c - 2a - c = 0 \longrightarrow c = 2a \nearrow \quad c = 2/5$

$\therefore \quad b + 2d + \frac{6}{5} = 0 \quad \& \quad d - 2b + \frac{2}{5} = 0$

$d = 2b - \frac{2}{5}$

$\therefore \quad b + 4b - \frac{4}{5} + \frac{6}{5} = 0$

$\rightarrow b = -\frac{2}{25}$

$d = -\frac{4}{25} - \frac{2}{5} = -\frac{14}{25}$ 4

$\therefore \quad y = Ae^{-x}\cos x + Be^{-x}\sin x$

$+ \left(\frac{1}{5}x - \frac{2}{25}\right)\cos x$

$+ \left(\frac{2}{5}x - \frac{14}{25}\right)\sin x.$ 1

10 altogether