First-order ODEs

(i) Separable, nonlinear equations: Separable means that the ODE can be written in the form,

$$\frac{dy}{dx} = f(x)g(y).$$

Solution:

$$\int \frac{dy}{g(y)} = C + \int f(x)dx,$$

where C is a constant of integration that is determined by an initial condition of the form, $y(x_0) = y_0$.

Example: $y' = (y+1)\sin x$. Solution:

$$\int \frac{dy}{y+1} = C + \int \sin x dx.$$

Integrating:

$$\ln|y+1| = C - \cos x, \qquad \text{or} \qquad y = \pm\exp(C - \cos x) - 1 \equiv ae^{-\cos x} - 1,$$

where a is a new constant of integration (and takes into account the $\pm$ which came from the absolution value within the logarithm).

Note that the absolute value is necessary inside the logarithm in order for the solution to make sense when $y < -1$.

(ii) Linear equations: Linear means that the ODE can be written in the form,

$$\frac{dy}{dx} + p(x)y = q(x).$$

The integrating factor,†

$$I(x) = \exp\left[\int^x p(\tilde{x})d\tilde{x}\right],$$

satisfies $dI/dx = pI$, and so multiplication by I turns the ODE into:

$$I\frac{dy}{dx} + pIy = I\frac{dy}{dx} + \frac{dI}{dx}y = \frac{d}{dx}(Iy) = qI.$$

Thus the solution is

$$y(x) = \frac{C}{I(x)} + \frac{1}{I(x)}\int^x I(\tilde{x})q(\tilde{x})d\tilde{x},$$

where C is again determined by the initial condition.

Example: $y' - y\sin x = \sin x$. ($p(x) = -\sin x$ and $q(x) = \sin x$.) Solution:

$$I = \exp\left[-\int \sin x \, dx\right] = e^{\cos x}.$$

Hence,

$$y = Ce^{-\cos x} + e^{-\cos x}\int^x e^{\cos\tilde{x}}\sin\tilde{x}\, d\tilde{x} = Ce^{-\cos x} - 1.$$

† Note the introduction of $\tilde{x}$ to distinguish the integration variable, and the notation $\int^x d\tilde{x}$ to indicate the integral "upto x".

Graphical Methods

The direction field for the equation $dy/dx = F(x, y)$:

- Place a grid on the $(x, y)-$plane.
- At each gridpoint, (x_n, y_n), estimate the slope of the solution, $[dy/dx]_{x=x_x, y=y_n} = F(x_n, y_n)$. Draw small line segments through each gridpoint with this slope.
- Thread curves through the grid so that the slope of the curve matches the slope of the line segments whenever it passes close to a gridpoint.
- If an initial condition is given, draw the curve passing through that particular point.

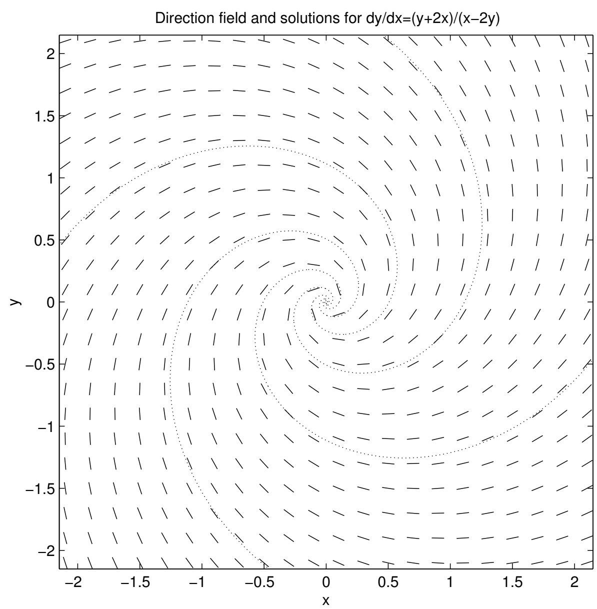

FIGURE 1. The direction field and sample solution curves for the first-order ODE, $dy/dx = (y + 2x)/(x - 2y)$. The graph is plotted using MATLAB; the line segments showing the direction field is constructed using the method described in ENG27L Assignment 2.

An alternative to the use of a grid is the so-called method of isoclines. Here one draws curves of constant slope (that is, let $s = dy/dx$ and solve $s = F(x, y)$ for $y(x, s)$; draw these curves on the plane), and then adds line segments with that slope passing through various points on these curves. Finally, one threads the solutions curves through the picture, again matching the slope with the line segments.

E. Kreyszig, Advanced Engineering Mathematics, Chapter 1.

Second-order, linear ODEs with constant coefficients

General form:

$$\begin{array}{rl} ay'' + by' + cy = 0 & \text{Homogeneous} \\ ay'' + by' + cy = f(x) & \text{Inhomogeneous.} \end{array}$$

Initial-value problem: $y(x)$ and $y'(x)$ are specified at a particular point, x_0. *e.g.* $y(x_0) = 1$ and $y'(x_0) = 1$.
Boundary-value problem: Conditions are applied on $y(x)$ and $y'(x)$ at two separate points, x_1 and x_2. e.g. $y(x_1) = 1$ and $y(x_2) = 1$.

Solution of the homogeneous problem: Pose $y = Ae^{mx}$, then we obtain the auxiliary equation,

$$am^2 + bm + c = 0, \qquad \text{or} \qquad m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(a) Real, unequal roots, m_1 and m_2: General Solution,

$$y(x) = Ae^{m_1 x} + Be^{m_2 x}.$$

(b) Real, equal roots, $m = m_1 = m_2$: General Solution,

$$y(x) = (Ax + b)e^{mx}.$$

(c) Complex roots, $m = \alpha \pm i\beta$: General Solution,

$$y(x) = \begin{cases} Ae^{(\alpha+i\beta)x} - Be^{(\alpha-i\beta)x} \\ e^{\alpha x}(C\cos\beta x + D\sin\beta x) \\ Re^{\alpha x}\cos(\beta x + \gamma) \end{cases}.$$

Examples:

- $y'' - 2y' - 3y = 0$. Auxiliary equation: $m^2 - 2m - 3 = (m+1)(m-3) = 0$, implying $m = -1$ or 3. Solution:

$$y(x) = Ae^{3x} + Be^{-x}$$

(with A and B arbitrary constants).

- $y'' - 2y' + y = 0$. Auxiliary equation: $m^2 - 2m + 1 = (m-1)^2 = 0$, implying $m = 1$. Solution:

$$y(x) = (ax + b)e^x.$$

(with a and b arbitrary constants).

- $y'' - 2y' + 5y = 0$. Auxiliary equation: $m^2 - 2m + 5 = 0$, or $(m-1)^2 = -4$, implying $m = 1 \pm 2i$. Solution:

$$y(x) = \begin{cases} Ae^{(1+2i)x} + Be^{(1-2i)x} \\ e^x(C\cos 2x + D\sin 2x) \\ Re^x\cos(2x + \gamma) \end{cases}$$

(with A, B, C, D, R and γ arbitrary constants).

Solution of the inhomogeneous problem: The general solution has the form of the homogeneous solutions plus a particular solution. General strategy:

(a) Find the homogeneous solutions (satisfying $ay'' + by' + cy = 0$)

(b) find a particular solution (by posing a trial solution based on $f(x)$ and containing constants to be determined)

(c) apply any initial/boundary conditions.

Inhomogeneous term, $f(x)$:	Trial particular solution (with constants C, D, ... to be determined):
$e^{\eta x}$	$Ce^{\eta x}$
Polynomial of degree n	Polynomial of degree n $(Cx^n + Dx^{n-1} + ...)$
$\cos\omega x$ or $\sin\omega x$	$C\cos\omega x + D\sin\omega x$
$e^{\eta x}\cos\omega x$ or $e^{\eta x}\sin\omega x$	$(C\cos\omega x + D\sin\omega x)e^{\eta x}$
(Polynomial of degree n)$e^{\eta x}$	$(Cx^n + Dx^{n-1} + ...)e^{\eta x}$
Homogeneous solution	$Cxf(x)$

One must beware of inhomogeneous terms that are also of the same form as a homogeneous solution. In this case, the trial solution will not work (the trial completely disappears from the left-hand side of the equation), and one must modify the guess by adding a further factor of x.

Examples:

- $y'' - 4y' + 3y = f(x)$, for $f(x) = e^{2x}$, $\cos x$, x^2 and e^x.

Homogeneous solutions: Auxilliary equation, $m^2 - 4m - 3 = (m-3)(m-1) = 0$, implying $m = 1$ or 3. Hence the homogeneous solutions are $Ae^x + Be^{3x}$, where A and B are arbitrary constants.

Particular solutions:

For $f(x) = e^{2x}$, try $y = ae^{2x}$. Plugging and chugging implies that this is a solution if $4a - 8a + 3a = 1$; that is, if $a = -1$.

For $f(x) = \cos x$, try $y = a\cos x + b\sin x$. Plug and chug; the solution works if $-a - 4b + 3a = 1$ and $-b + 4a + 3b = 0$, which gives $a = 1/10$ and $b = -1/5$.

For $f(x) = x^2$, try $y = ax^2 + bx + c$. Plug and chug to find that $3a = 1$, $3b - 8a = 0$ and $3c - 4b + 2a = 0$, or $a = 1/3$, $b = 8/9$ and $c = 26/27$.

For $f(x) = e^x$, the trial solution $y = ae^x$ will not work because the inhomogeneous term has the form of a homogeneous solution. Now we must try the solution axe^x. Plugging and chugging implies that $-2a = 1$, or $a = -1/2$.

Finally, if there are initial or boundary conditions, apply them to find A and B.

E. Kreyszig, Advanced Engineering Mathematics, Chapter 2.

Matrices and determinants

An $m \times n$ matrix is an ordered array of numbers arranged in m rows and n columns. The entries or elements are denoted a_{jk}, where the subscripts, j and k, denote the row and the column in which the element is located:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$e.g. \qquad \begin{pmatrix} 2 & -5 & 1/3 \\ \pi & 0 & 0.7 \end{pmatrix} \quad \text{is} \quad 2 \times 3 \qquad \begin{pmatrix} 1+i \\ \sqrt{3} \\ -i \end{pmatrix} \quad \text{is} \quad 3 \times 1$$

Special matrices

- A matrix with a single column (row) is also called a column (row) vector.
- A matrix with as many rows as columns ($m = n$) is called a "square matrix". The diagonal from top-left to bottom-right is called the "leading diagonal", and consists of the elements, a_{11}, a_{22}, a_{33}, ..., a_{nn}.
- A symmetric matrix is a square matrix whose element in the j^{th} row and k^{th} column is equal to the element in the k^{th} row and j^{th} column. *i.e.* $a_{jk} = a_{kj}$ for all possible j and k. The matrix is symmetric on "reflection" about the leading diagonal.

$$e.g. \quad \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 3 & 4 & -7 \\ -2 & 4 & -1 & 6 \\ 3 & -7 & 6 & 2 \end{pmatrix}.$$

- If $a_{jk} = -a_{kj}$ for all possible j and k, the square matrix A is "anti-symmetric", or "skew-symmetric". Since $a_{jj} = -a_{jj}$, the elements on the leading diagonal must all vanish.

$$e.g. \quad \begin{pmatrix} 0 & -1 & -2 & 3 \\ 1 & 0 & 4 & -7 \\ 2 & -4 & 0 & 6 \\ -3 & 7 & -6 & 0 \end{pmatrix}.$$

- A square matrix is "upper (lower) triangular" if all elements below (above) the leading diagonal are zero.

$$e.g. \quad \text{Upper triangular :} \quad \begin{pmatrix} -1 & -2 & 3 \\ 0 & 4 & -7 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{Lower triangular :} \quad \begin{pmatrix} -1 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 8 & 1 \end{pmatrix}.$$

- A "diagonal matrix" is a square matrix with zeros everywhere but on leading diagonal.

$$e.g. \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- The "identity" or "unit" matrix of order $n \times n$ is the diagonal matrix of order $n \times n$ in which all the diagonal elements are equal to unity. It is denoted $I_{n\times n}$ or I_n, or simply I.

$$e.g. \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- The zero matrix of order $m \times n$ is denoted $O_{m\times n}$ and has zero for each element.

Basic rules of matrix algebra

• Equality: The matrices A and B are equal if and only if all corresponding elements are equal and the orders are the same: $a_{jk} = b_{jk}$ for all possible j and k.

• Addition and subtraction: The matrices A and B can be added and subtracted only if they have the same order. Their sum $A + B$ and difference $A - B$ have the elements, $a_{jk} + b_{jk}$ and $a_{jk} - b_{jk}$, respectively.

• Multiplication by a scalar: The matrix λA, where λ is a scalar (number), has elements λa_{jk}.

• Matrix multiplication: The product $AB = C$ of two matrices A and B exists if the number of columns of A equals the number of rows of B (*i.e.* A is order $m \times n$ and B is order $n \times p$). The matrix C has elements given by $c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk}$. *i.e.* the elements at the intersection of the j^{th} row and the k^{th} column is the sum of products of elements from the j^{th} row of A and the k^{th} column of B, taken in order:

$$\begin{pmatrix} & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ \cdot & \cdot & c_{jk} & \cdot \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \end{pmatrix} = \begin{pmatrix} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \\ \end{pmatrix} \begin{pmatrix} b_{1k} \\ b_{2k} \\ \cdot \\ \cdot \\ \cdot \\ b_{nk} \end{pmatrix}$$

The product, AB, has as many rows as A, and as many columns as B (*i.e.* AB has order $m \times p$ if A is $m \times n$ and B is $n \times p$). Even when AB exists, it is not necessarily the case that BA exists, and if it does exist, it is not necessarily equal to AB.

• Provided all the sums and products exist, $A + B = B + A$, and $(A + B)C = AC + BC = BC + AC = (B + A)C$.

• Provided the products exist, $(AB)C = A(BC) = ABC$.

• For a square matrix, $A^2 = AA$, $A^3 = AAA$, and $A^p = AAA...A$ (p factors of A).

• Multiplication by the identity or unit matrix: If A is $m \times n$, $I_m A = A I_n = A$.

• Transpose: The transpose of the $m \times n$ matrix A, denoted A^T (or sometimes A'), is the $n \times m$ matrix obtained by interchanging the rows and columns of A. Thus, the elements of A^T are a_{kj}.

• Transpose of a product: If AB exists, $(AB)^T = B^T A^T$.

The matrix inverse

(a) Matrix inverse: A square matrix A may possess an inverse, denoted A^{-1}, with the properties, $AA^{-1} = A^{-1}A = I$. Consequently, the linear system, $Ax = b$, may be solved as $x = A^{-1}b$.

(b) Inverse of a product: $(AB)^{-1} = B^{-1}A^{-1}$.

(c) Inverse of a power: $(A^p)^{-1} = (A^{-1})^p = A^{-p}$, for p a positive integer.

(d) $(A^T)^{-1} = (A^{-1})^T$.

(e) The inverse of a diagonal matrix with elements, a_{11}, a_{22}, ..., a_{nn}, is the diagonal matrix with elements, a_{11}^{-1}, a_{22}^{-1}, ..., a_{nn}^{-1}.

(f) The inverse of a lower (upper) triangular matrix is an lower (upper) triangular matrix.

(g) The inverse of a symmetric matrix is also symmetric.

(h) The inverse A^{-1} exists if and only if the determinant of A is not equal zero. If det $A = 0$, the inverse does not exist and the matrix is called "singular".

E. Kreyszig, Advanced Engineering Mathematics, Chapter 6.

Determinants

An n^{th}-order determinant is written as:

$$\begin{vmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{vmatrix}.$$

Its value is a scalar (number), and is fomulated in terms a set of products of the elements, a_{jk}.

The inverse of a 2×2 *matrix:*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \qquad \det A = a_{11}a_{22} - a_{12}a_{21}.$$

Minors and Cofactors: If the j^{th} row and k^{th} column of the determinant are deleted, a determinant of order $n-1$ is obtained, and is called the "minor" of a_{jk}, and denoted M_{jk}. The "cofactor", A_{jk}, of the element, a_{jk}, is $A_{jk} = (-1)^{j+k} M_{jk}$.

Expansion of a determinant: The formula,

$$\det A = a_{11}A_{11} + a_{12}A_{12} + ... + a_{1n}A_{1n},$$

expresses a determinant in terms of determinants of lower order, using the top row. The reduction can be performed using any row or column:

$$\det A = \sum_{k=1}^{n} a_{jk}A_{jk} \qquad (j \text{ fixed; expanding using the } j^{th} \text{ row}),$$

$$\det A = \sum_{j=1}^{n} a_{jk}A_{jk} \qquad (k \text{ fixed; expanding using the } k^{th} \text{ column}).$$

Properties of determinants:

(a) The determinant of a diagonal or triangular matrix is the product of the elements on the leading diagonal. e.g.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 12 = \begin{vmatrix} 1 & 6 & 9 \\ 0 & 6 & 4 \\ 0 & 0 & 2 \end{vmatrix}.$$

(b) Interchange of two rows (columns) reverses the sign of the determinant. e.g.

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}.$$

(c) When all elements of one row (column) are multiplied by a scalar, k, the determinant is multiplied by k. e.g.

$$2\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = 4\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}.$$

(d) If all elements of any row (column) are zero the value of the determinant is zero. e.g.

$$\begin{vmatrix} 4 & 2 \\ 0 & 0 \end{vmatrix} = 0.$$

(e) If two rows (columns) are identical, the value of the determinant is zero. e.g.

$$\begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} = 0.$$

(*f*) If one row (column) is a scalar multiple of another row (column), the value of the determinant is zero. e.g.

$$\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 0.$$

(*g*) When any multiple of one row (column) is added to another row (column), the value of the determinant is unchanged. e.g.

$$\begin{vmatrix} 4 & 2 \\ 4 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 0 & 4 \end{vmatrix}$$

(row 2 is replaced by row 2 – row 1).

(*h*) $\det A^T = \det A$. e.g.

$$\begin{vmatrix} 4 & 2 \\ 4 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 4 \\ 2 & 6 \end{vmatrix}.$$

(*i*) $\det (AB) = \det A \times \det B = \det (BA)$.

(*j*) $\det (A^{-1}) = (\det A)^{-1}$.

Worked example

By expansion (using the top row, since there is a zero in it):

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1 \times \begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= (6 \times 2 - 4 \times 0) - 3(2 \times 2 - (-1) \times 4) = 12 - 3 \times 8 = -12.$$

By row/column manipulation

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 4 \\ -1 & 0 & 2 \end{vmatrix} \quad \text{(row 2)} \rightarrow \text{(row 2)} - 2 \times \text{(row 1)}$$

$$= 4 \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} \quad \text{take out factor of 4 from row 2}$$

$$= 4 \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{vmatrix} \quad \text{(row 3)} \rightarrow \text{(row 3)} - 2 \times \text{(row 2)}$$

$$= 4 \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{vmatrix} \quad \text{(row 1)} \rightarrow \text{(row 1)} + \text{(row 3)}$$

$$= -4 \begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{vmatrix} \quad \text{exchange rows 1 and 3} \qquad = 4 \begin{vmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{exchange rows 2 and 3}$$

$$= 4 \times (-1) \times 3 \times 1 = -12 \qquad \text{(using the properties of a diagonal matrix).}$$

E. Kreyszig, Advanced Engineering Mathematics, Chapter 6.

Eigenvalues and eigenvectors

The eigenvalues, λ, and eigenvectors, x, of an $n \times n$ matrix, A, are both defined by the equation,

$$Ax = \lambda x$$

The eigenvalue equation is written in the alternative form,

$$(A - \lambda I)x = 0;$$

nontrivial solutions exist if

$$\det(A - \lambda I) = 0,$$

which determines λ. The eigenvectors are then the (non-unique) solutions of $(A - \lambda I)x = 0$, for each possible value of λ.

The "characteristic polynomial" of A is the expansion of the determinant of $A - \lambda I$, expressed as a polynomial in λ:

$$\det(A - \lambda I) \equiv \det(A) + a_1\lambda + a_2\lambda^2 + ... + a_{n-1}\lambda^{n-1} + a_n\lambda^n.$$

Factorizing:

$$\det(A - \lambda I) \equiv (\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda),$$

where λ_j, with $j = 1, 2, ..., n$, are the eigenvalues, and $\det(A) = \lambda_1\lambda_2...\lambda_n$.

e.g. Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

The eigenvalues are the solution of

$$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0.$$

That is,

$$-\lambda^3 + 2\lambda^2 + 4\lambda - 8 = (-2 - \lambda)(2 - \lambda)^2 = 0,$$

or $\lambda = -2$ or 2.

For $\lambda = -2$, the eigenvector satisfies

$$\begin{pmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that

$$x = \alpha \begin{pmatrix} 4 \\ 1 \\ -7 \end{pmatrix},$$

with α an arbitrary constant.

For $\lambda = 2$, the eigenvector satisfies

$$\begin{pmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that

$$x = \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

with β another arbitrary constant.

• Real matrices (matrices with entries that are purely real) may have real eigenvalues, or eigenvalues that occur as complex conjugate pairs, or a combination of both.

• Symmetrical matrices have purely real eigenvalues.

• The eigenvalues of A^p are λ^p, for p a positive integer, if λ denotes the eigenvalues of A. The eigenvectors of A are also the eigenvectors of A^p.

• If A^{-1} exists, the eigenvalues of A^{-p} are λ^{-p}, for p a positive integer. The eigenvectors of A are also the eigenvectors of A^{-p}.

• If $\det(A) = 0$, at least one of the eigenvalues must equal zero.

• The eigenvalues of a triangular (diagonal) matrix are equal to the diagonal elements of the matrix. *i.e.* $\lambda_j = A_{jj}$. *e.g.*

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

has eigenvalues 1, 4 and 6. The corresponding eigenvectors are

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \beta \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \qquad \gamma \begin{pmatrix} 16 \\ 25 \\ 10 \end{pmatrix},$$

where α, β and γ are arbitrary constants.

E. Kreyszig, Advanced Engineering Mathematics, Chapter 7.

Systems of ordinary differential equations

A system of n ordinary differential equations, such as

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + f_1(t) \\ x_2' &= a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + f_2(t) \\ &... \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n + f_n(t) \end{aligned}$$

can be written in the matrix form,

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t).$$

The system need not always have first derivatives – *e.g.* $\mathbf{x}'' = A\mathbf{x} + \mathbf{f}$.

To find the homogeneous solution, pose $\mathbf{x} = \mathbf{v}e^{\lambda t}$. Then,

$$\lambda\mathbf{v} = A\mathbf{v}.$$

Hence, there are n values for λ given by the eigenvalues of A; the corresponding eigenvectors give $\mathbf{v}$. *e.g.* Consider

$$\mathbf{x}' = A\mathbf{x} \equiv \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \mathbf{x}.$$

The eigenvalues of A are -1 and 4. The corresponding eigenvectors are

$$A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (\lambda = -1) \qquad \text{and} \qquad A_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (\lambda = 4),$$

where A_1 and A_2 are arbitrary. The (homogeneous) solution is therefore

$$\mathbf{x} = A_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + A_2 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

For the particular solution, $\mathbf{x}_p$ say, we pose a trial based on the form of $\mathbf{f}(t)$. *e.g.* $\mathbf{f}(t) = \mathbf{f}_0 e^{2t}$, pose $\mathbf{x}_p = \mathbf{u}e^{2t}$, where $\mathbf{u}$ is a constant vector to be determined by substitution into the system of ODEs:

$$\mathbf{x}_p' = A\mathbf{x}_p + e^{2t}\mathbf{f}_0 \quad \longrightarrow \quad \mathbf{u} = (2I - A)^{-1}\mathbf{f}_0.$$

The general solution is the combination of the homogeneous and particular solutions:

$$\mathbf{x} = A_1\mathbf{v}_1 e^{\lambda_1 t} + A_2\mathbf{v}_2 e^{\lambda_2 t} + ... + A_n\mathbf{v}_n e^{\lambda_n t} + \mathbf{x}_p,$$

where the constants $A_1, ..., A_n$ are to be determined by imposition of the initial condition. Note that the computation of the eigenvectors $\mathbf{v}_k$ is not unique – there is an overall undetermined constant. This constant can be taken to be the corresponding constant A_k.

E. Kreyszig, Advanced Engineering Mathematics, Chapter 3.

Laplace Transforms

The definition:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \equiv \bar{f}(s).$$

There is an *inverse transform*,

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \int_C e^{st} \bar{f}(s) \frac{ds}{2\pi} \equiv f(t),$$

where C is a particular curve (the "Bromwich contour") on the complex s plane. (The inverse transform cannot be used without more knowledge of complex analysis.)

Being defined by an integral, the transform has the linearity property:

$$\mathcal{L}\{af(t) + bg(t)\} = \mathcal{L}\{af(t)\} + \mathcal{L}\{bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} = a\bar{f}(s) + b\bar{g}(s).$$

To solve ODEs with Laplace Transforms: Note the relations,

$$\mathcal{L}\{y\} = \bar{y}(s) \qquad \mathcal{L}\{y'\} = s\bar{y}(s) - y(0) \qquad \mathcal{L}\{y''\} = s^2\bar{y}(s) - y'(0) - sy(0),$$

which follow from the definition of the transform and integrating by parts. Hence, when we Laplace transform the ODE,

$$ay'' + by' + cy = f(t),$$

we obtain

$$\bar{y}(s) = \frac{\bar{f}(s) + ay'(0) + (as + b)y(0)}{as^2 + bs + c}.$$

The final terms in the numerator are given by the initial conditions, and it is convenient to apply them at this stage. Finally, and most difficultly, we need to invert the Laplace transform so that we may recover $y(t)$ from $y(s)$. This is to be accomplished using some algebraic tricks and the table of transforms.

Some useful tricks:

- Partial fractions: *e.g.*

$$\frac{1}{(s-d)(s-e)} = \frac{A}{s-d} + \frac{B}{s-e},$$

$$\frac{1}{(s-d)^2(s-e)} = \frac{A}{(s-d)^2} + \frac{B}{s-d} + \frac{C}{s-e}$$

and

$$\frac{1}{(s^2+d^2)(s-e)} = \frac{As+B}{s^2+d^2} + \frac{C}{s-e},$$

where A, B and C can be determined by recombining the fractions and equating coefficients of the powers of s. The rule is, add as many powers of s in the numerator (a polynomial in s) as needed – one less than in the denominator.

- First shifting theorem: We know $\mathcal{L}\{e^{at}f(t)\} = \bar{f}(s-a)$. If $\bar{y}(s)$ is not in the table but contains s in the combination $s-a$, then $\bar{y}(s-a)$ might be in the table, and we can use a shift to invert the transform. *e.g.* $1/(s+4)^4$ is not in the table, but $1/s^2$ is, and would have an inverse transform of $t^3/3!$. Comparing $s+4$ with $s-a$ implies that $a = -4$. Hence we use the shifting theorem and write $\mathcal{L}^{-1}\{1/(s+4)^4\} = e^{-4t}t^3/3!$. Another example:

$$\frac{s}{s^2+2s+5} = \frac{s}{(s+1)^2+4} = \frac{(s+1)}{(s+1)^2+4} - \frac{1}{(s+1)^2+4} = \mathcal{L}\{e^{-t}\cos 2t\} - \frac{1}{2}\mathcal{L}\{e^{-t}\sin 2t\}.$$

E. Kreyszig, Advanced Engineering Mathematics, Chapter 5.

Table of useful Laplace Transforms

$f(t)$	$\bar{f}(s)$
1	$1/s, \quad s > 0$
$t^n, \quad n = 0, 1, 2, ...$	$n!/s^{n+1}, \quad s > 0$
e^{at}	$1/(s-a), \quad s > a$
$\sin at$	$a/(s^2+a^2), \quad s > 0$
$\cos at$	$s/(s^2+a^2), \quad s > 0$
$\sinh at$	$a/(s^2-a^2), \quad s > \|a\|$
$\cosh at$	$s/(s^2-a^2), \quad s > \|a\|$
$t \sin at$	$2as/(s^2+a^2)^2, \quad s > 0$
$t \cos at$	$(s^2-a^2)/(s^2+a^2)^2, \quad s > 0$
First derivative: $y'(t)$	$s\bar{y}(s) - y(0)$
Second derivative: $y''(t)$	$s^2\bar{y}(s) - y'(0) - sy(0)$
Heaviside step function: $H(t-a)$	e^{-as}/s
Dirac delta function: $\delta(t-a)$	e^{-as}
First shifting theorem: $e^{at}f(t)$	$\bar{f}(s-a)$
e.g. $t^n e^{at}$	$n!/(s-a)^{n+1}$
Second shifting theorem: $f(t-a)H(t-a)$	$e^{-as}\bar{f}(s)$
e.g. $e^{6(t-a)}H(t-a)$	$e^{-as}/(s-6)$
Convolution theorem: $\int_0^t f(u)g(t-u)du$	$\bar{f}(s)\bar{g}(s)$