Definitions, terminology, notation...

ODEs: one seeks a function of a single variable (*e.g.* $y(x)$) that satisfies a differential equation – a given relation between the function and its derivatives (y, y', y'', ...).

PDEs: one seeks a function of multiple variables (*e.g.* $u(x,t)$) that satisfies a relation between that function and its partial derivatives.

Sample ODE; $y' = 2y + e^x$
Sample PDE: $u_t = u_{xx}$ (the heat or diffusion equation; subscripts used as shorthand for partial derivatives)

An ODE or PDE is LINEAR if the differential equation is a linear combination of the function and its derivatives. *i.e.* for an ODE in which $y(x)$ is related to its first derivative y', the equation is linear if it has the form

$$c_0 + c_1 y + c_2 y' = 0;$$

for an ODE in which y is related to both its first and second derivative, the equation is linear if

$$c_0 + c_1 y + c_2 y' + c_3 y'' = 0.$$

Here, c_0, c_1, ..., are either arbitrary functions of x or constants.

ORDER of an ODE: pick out the highest derivative of $y(x)$ in the ODE. If n is the number of derivatives, then the order of the ODE is also n.

e.g. $x^2 + y + yy' = 0$ is nonlinear and first order; $2x^2 + y + e^x y' + 3y'' = 0$ is linear and second order.

We focus on two types of first-order ODEs: linear and separable. The tricks to attack these types of ODEs are to cleverly rearrange the equations and find the solutions by computing two integrals.

Integrating factors

Consider the linear, first-order ODE

$$y' + p(x)y = q(x)$$

(for the general linear ODE, $c_0 + c_1 y + c_2 y' = 0$, this simply means that $p = c_1/c_2$ and $q = -c_0/c_2$). Let

$$I(x) = \exp\left(\int p dx\right) \quad \longrightarrow \quad \frac{dI}{dx} = pI$$

Multiply the ODE by the "integrating factor" $I(x)$:

$$qI = Iy' + (pI)y = Iy' + I'y = \frac{d}{dx}(Iy)$$

Hence

$$Iy = \int qI dx + C$$

where C is an arbitrary constant of integration, and so

$$y = \frac{1}{I}\int qI dx + \frac{C}{I}$$

e.g. $y' = 2y + e^x$. We have $I = \exp\int(-2)dx = e^{-2x}$ and $\int qI dx = \int e^x e^{-2x} dx = -e^{-x}$, so $y = Ce^{2x} - e^x$.

e.g. $y' = 2xy$. We have $I = \exp\int(-2x)dx = e^{-x^2}$ and $\int qI dx = 0$, so $y = Ce^{x^2}$.

e.g. $y' = 2xy^2 + 3x$ is not linear, silly!

N.B. The solution is not unique given that C is arbitrary!

Separable first-order ODE

A first-order ODE is separable if it can be written in the form

$$y' = f(x)g(y)$$

i.e. the dependence on x and y can be divided up into two factors. The ODE need not be linear; indeed, it is most often nonlinear.

e.g. $y' = 2xy$ ($f = 2x$, $g = y$) or $y' = 2xe^{-y}$ ($f = 2x$, $g = e^{-y}$).

Solution strategy: rewrite the ODE and then integrate...

$$f(x) = \frac{y'}{g(y)} \quad \longrightarrow \quad \int f(x)dx + C = \int \frac{dy}{dx}\frac{dx}{g(y)} = \int \frac{dy}{g(y)},$$

where C is another integration constant. At this stage, since $f(x)$ and $g(y)$ are known functions with computable integrals, we're largely done.

e.g. $y' = 2xy$ ($f = 2x$, $g = y$). So we have $\log|y| = C + x^2$. Exponentiating gives $|y| = e^C e^{x^2}$, and so $y = \pm e^C e^{x^2}$. Let $\hat{C} = \pm e^C$ (another arbitrary constant), giving $y = \hat{C}e^{x^2}$.

e.g. $y' = 2xe^{-y}$ ($f = 2x$, $g = e^{-y}$). Now, $e^y = C + x^2$, and so $y = \log(C + x^2)$.

Making the solution unique (fixing C)

To fix the arbitrary constant of integration, we need an additional condition - a starting value or initial condition, of the form $y(a) = y_0$ for some given a and y_0.

e.g. $y' = 2xy$ with $y(0) = 1$. The solution we get by either noting that this ODE is either linear or separable is $y = Ce^{x^2}$. Putting $x = 0$ gives $1 = C$, so $y = e^{x^2}$.

e.g. $y' = 2xe^{-y}$ with $y(0) = 0$. The solution we get from noting that this is a separable ODE is $y = \log(C + x^2)$. Putting $x = 0$ gives $0 = \log C$, or $C = 1$. Hence $y = \log(1 + x^2)$.

Worked problem: Vincent Peter Lovelace is out jogging one morning in his fancy new running shoes. He uses his leg muscles to apply thrust to push himself forward with a force $Ae^{-\alpha v}$, where $v(t)$ is his speed at time t and A and α are constants. His progress is resisted by his overly tight running shorts, of strength εA with $0 < \varepsilon < 1$. Vincent's mass is m and he starts from a standstill ($v(0) = 0$). Determine Vincent's speed. What if Vincent's shorts were stretching as he jogged and the resistance was $\varepsilon A/(1 + t)$?

Newton's law states that force equals mass times acceleration, so

$$m\dot{v} = A(e^{-\alpha v} - \varepsilon).$$

This is a nonlinear separable ODE, for which

$$\frac{A}{m}\int dt = \int \frac{e^{\alpha v}dv}{1 - \varepsilon e^{\alpha v}} \equiv \frac{1}{\alpha\varepsilon}\int \frac{dz}{1 - z}$$

where $z = \varepsilon e^{\alpha v}$. Hence

$$\frac{\alpha\varepsilon At}{m} = C - \log|1 - \varepsilon e^{\alpha v}|$$

for some arbitrary constant C. But if $v(0) = 0$, then

$$v(t) = \frac{1}{a}\log\left[\frac{1 - (1 - \varepsilon)e^{-\varepsilon\alpha At/m}}{\varepsilon}\right]$$

If the shorts were stretching during the jog, the ODE becomes neither linear nor separable,

$$m\dot{v} = A\left(e^{-\alpha v} - \frac{\varepsilon}{1 + t}\right),$$

so we'd better panic. Fortunately, the transformation $z = \varepsilon e^{\alpha v}$ places the ODE in the linear form,

$$\dot{z} + \frac{\gamma z}{1 + t} = \gamma \qquad \longrightarrow \qquad z(t) = \frac{\gamma(1 + t)}{\gamma + 1} + \left(\varepsilon - \frac{\gamma}{\gamma + 1}\right)(1 + t)^{-\gamma}, \qquad \text{with } \gamma = \frac{\varepsilon\alpha A}{m}.$$