Numerical/Graphical Methods

The definition of a derivative:

$$\frac{dy}{dx} = \mathrm{Lim}_{\epsilon\to 0}\frac{y(x+\epsilon)-y(x)}{\epsilon}$$

But from the ODE we also know that $dy/dx = F(x,y)$. Hence, taking a small but finite value of ϵ we arrive at the approximation

$$y(x+\epsilon) \approx y(x) + \epsilon F(x,y).$$

Given a starting condition, $y(a) = y_0$, we may now step away from $x = a$ to $x = a+\epsilon$, and calcuate $y(a+\epsilon)$, and then repeat the construction to continue to $x = a+2\epsilon$, $a+3\epsilon$ and so on. We thereby build an approximate solution curve. Evidently, each time a different starting condition is used, a different curve will be generated (the non-uniqueness of the general solution is removed by picking a starting condition).

The approximation above is the simplest type of numerical scheme to solve a differential equation by "finite differencing" (*i.e.* replacing the derivatives by differences).

An example is shown in the figure (red dots) for the ODE

$$\frac{dy}{dx} = (1-y)\cos x \qquad \left(\ y(x) = Ce^{-\sin x}+1\ \right)$$

and specific starting condition $y(0) = 0$. The figure also plots more solution curves with different starting values (as indicated by the green stars).

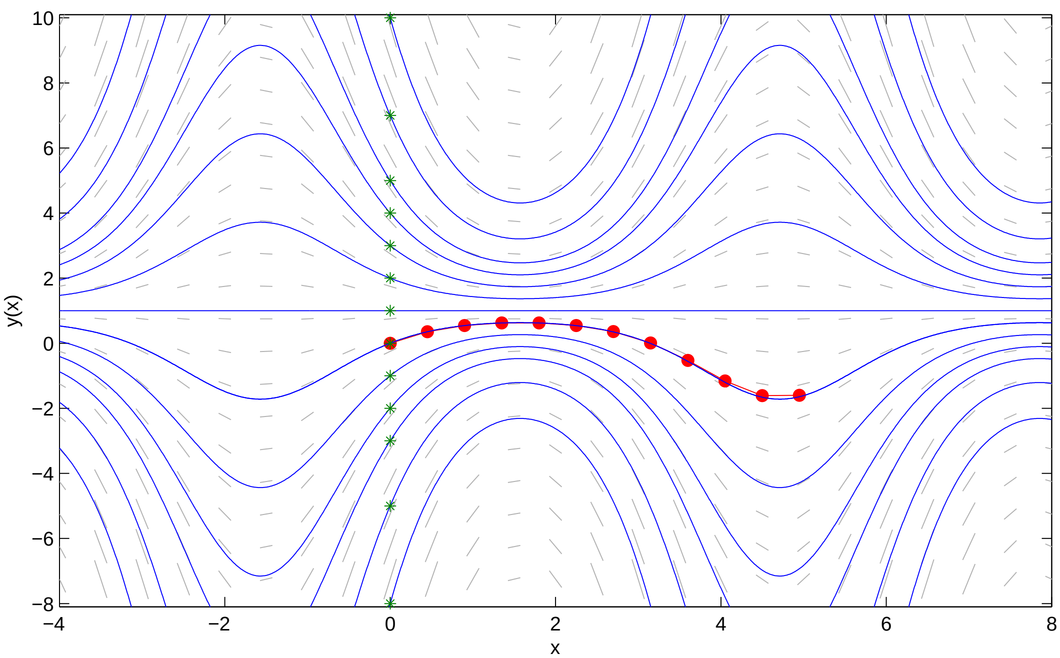

FIGURE 1. Solution curves for the first-order ODE, $dy/dx = (1-y)\cos x$, and the starting values at $x = 0$ indicated by stars. The short grey lines sample the direction field of the ODE.

A related method that seeks to plot qualitatively entire sets of solution curves exploits the "direction field" of the ODE: on the $(x,y)-$plane we place a grid, and then compute the slope of the solution at the grid points using the ODE $dy/dx = F(x,y)$. Short lines or arrows can then be plotted at those points to sample the direction field (see the figure). When a solution curve passes close to a grid point, its slope must match the line/arrow there. One can therefore thread curves through the grid to build a qualitative picture of the different solution branches.

A second example, for the ODE $y' = (1-y^2)\cos x$ is shown below (the exact solution of this separable ODE is $y = (Ce^{2\sin x}-1)/(Ce^{2\sin x}+1)$).

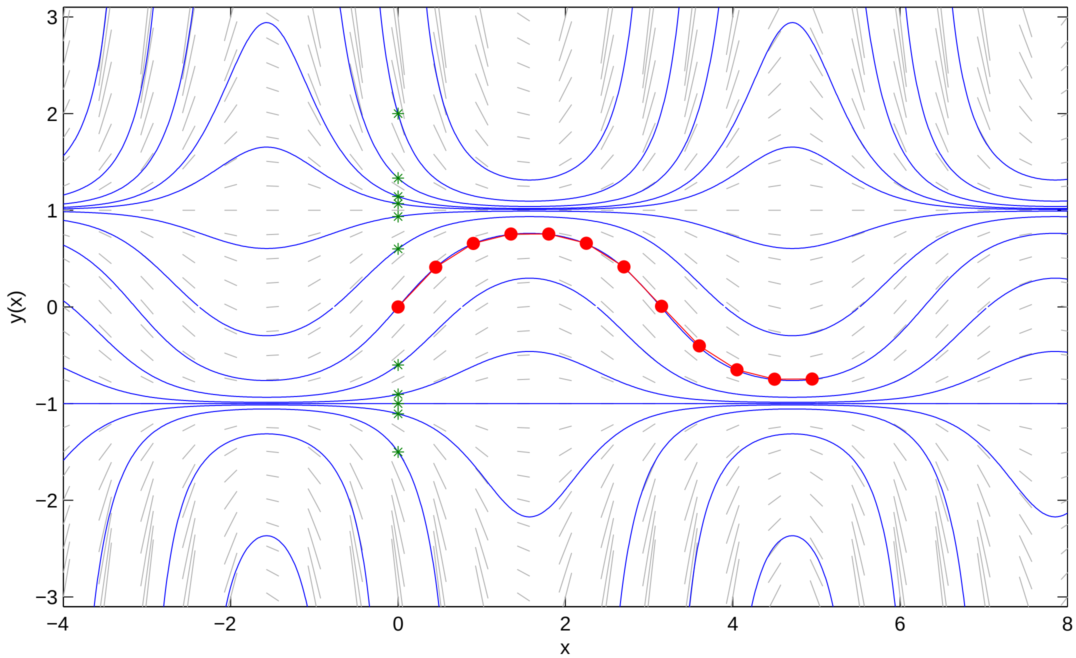

FIGURE 2. Solution curves for the first-order ODE, $dy/dx = (1 - y^2)\cos x$, and the starting values at $x = 0$ indicated by stars. The short grey lines sample the direction field of the ODE.

Second-order, linear ODEs with constant coefficients

General form: for constants a, b and c,

$$\begin{aligned} ay'' + by' + cy &= 0 \qquad \text{Homogeneous} \\ ay'' + by' + cy &= f(x) \quad \text{Inhomogeneous,} \end{aligned}$$

with $f(x)$ a prescribed function.

Strategy for the homogeneous problem: pose $y(x) = Ae^{mx}$, where A and m are constants. Since $y' = my$ and $y'' = m^2 y$ we have

$$(am^2 + bm + c)y = 0.$$

The choice $y = 0$ to solve this equation is trivial and uninteresting. Instead, we demand that the solution satisfy the "auxiliary equation",

$$am^2 + bm + c = 0, \qquad \text{or} \qquad m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with two roots $m = m_1$ and $m = m_2$. Thus, we arrive at the general solution

$$y(x) = A_1 e^{m_1 x} + A_2 e^{m_2 x},$$

for two different arbitrary constants A_1 and A_2 (making them different ensures the most general answer).

In many examples, the two solutions for m are real and unequal, but this is not always the case. To fix the constants A_1 and A_2 we now require **two** additional conditions.

• In an **Initial-Value Problem**, $y(x)$ and $y'(x)$ are specified at a particular point, x_0. *e.g.* $y(x_0) = 1$ and $y'(x_0) = 1$.

• For a **Boundary-Value Problem**, conditions are applied on either $y(x)$ or $y'(x)$ or a combo of both, at two separate points, x_1 and x_2. *e.g.* $y(x_1) = 1$ and $y(x_2) = 1$.

e.g. $2y'' - y' - 3y = 0, \quad \longrightarrow \quad 2m^2 - m - 3 = (2m-3)(m+1) = 0 \quad \longrightarrow \quad y(x) = A_1 e^{3x/2} + A_2 e^{-x}$

If $y(0) = 1$ and $y'(0) = 0$ (an initial-value problem), we have $A_1 + A_2 = 1$ and $3A_1/2 - A_2 = 0$, implying $A_1 = 2/5$ and $A_2 = 3/5$.

For $y(0) = 1$ and $y'(1) = 0$ (a boundary-value problem) we have $A_1 + A_2 = 1$ and $3A_1/2e^{3/2} - A_2 e^{-1} = 0$, *etc.*

Complex solutions to the auxiliary equation

If the auxiliary equation has complex solutions ($b^2 < 4ac$) then $m = \alpha \pm i\beta$ for $\alpha = -b/(2a)$ and $\beta = \sqrt{4ac - b^2}/(2a)$.

One can persevere with these complex solutions and again write

$$y(x) = A_1 e^{m_1 x} + A_2 e^{m_2 x} = e^{\alpha x}(A_1 e^{i\beta x} + A_2 e^{-i\beta x}),$$

but the arbitrary constants are likely to turn out to be complex in any given initial or boundary-value problem. Moreover, one must work with the complex exponentials. Alternatively one may exploit Euler ($e^{i\beta} = \cos\beta + i\sin\beta$) to rewrite the general solution so that it takes a purely real form:

$$m = \alpha \pm i\beta \qquad \longrightarrow \qquad y(x) = e^{\alpha x}(C\cos\beta x + D\sin\beta x)$$

for two other arbirary constants C and D. With the clever use of some trig relations, we can even write

$$y(x) = Re^{\alpha x}\cos(\beta x + \gamma)$$

for two new arbitary constants R and γ. When satisfying the two additional conditions in an initial or boundary-value problem, real values of y and/or y' are typically provided, ensuring that C, D, R and γ turn out to be real numbers.

$$e.g. \quad y'' + 4y = 0, \qquad \longrightarrow \qquad m^2 + 4 = (m - 2i)(m + 2i) = 0,$$

giving

$$y(x) = \begin{cases} A_1 e^{2ix} + A_2 e^{-2ix} \\ C\cos 2x + D\sin 2x \\ R\cos(2x + \gamma) \end{cases}$$

In the initial-value problem, $y(0) = 0$ and $y'(0) = 2$ we obtain $y(x) = \sin 2x$ (the conditions give, for example, $C = 0$ and $2D = 2$).

For the boundary-value problem, $y(0) = 0$ and $y(\pi/4) = 1$, we find the same solution (the conditions give, for example, $C = 0$ and $D\sin(\pi/2) = 1$).

Equal roots to the auxiliary equation

If $b^2 = 4ac$, the auxiliary equation has the real equal roots $m = -b/(2a)$ suggesting that $y(x) = Ae^{mx}$ alone. However, two different solutions are needed in order to formulate the general solution. To find another solution we use the following trick (*Reduction of Order*): let $y(x) = A(x)e^{mx}$. Plugging this into the ODE:

$$ay'' + by' + cy = (am^2 + bm + c)Ae^{mx} + a(2mA' + A'')e^{mx} + bA'e^{mx} = 0$$

But $am^2 + bm + c = 0$ and $2ma = -b$. Thus $A'' = 0$, implying $A = B + Cx$ and

$$y(x) = (B + Cx)e^{mx}.$$

i.e. the first term is the original solution; the second is the needed new (and different) solution.

$$e.g. \quad y'' + 4y' + 4y = 0, \qquad \longrightarrow \qquad m^2 + 4m + 4 = (m + 2)^2 = 0, \qquad \longrightarrow \qquad y(x) = (B + Cx)e^{-2x}.$$

If $y(0) = 0$ and $y'(0) = 1$, then $B = 0$ and $C = 1$, giving $y(x) = xe^{-2x}$.