Laplace transforms

The Laplace transform is defined by the integral

$$\mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t) dt = \overline{y}(s)$$

Within the integral a new variable s appears. Thus the transform is a function of s; we add the bar above the original function symbol to denote the new function of s.

Sample Laplace transforms: Using the definition, $\mathcal{L}\{1\} = \frac{1}{s}$ provided Real$(s) > 0$, and $\mathcal{L}\{e^{mt}\} = \frac{1}{s-m}$ provided Real$(s) > m$.

The transform domain: Note that the transform is only defined (*i.e.* the integral is finite) for certain ranges of s, which could be complex.

Linearity: the transform is linear (*i.e.* acts upon the function itself, rather than a power of it or some such thing), which implies

$$\mathcal{L}\{Ay_1(t) + By_2(t)\} = A\mathcal{L}\{y_1(t)\} + B\mathcal{L}\{y_2(t)\} = A\overline{y}_1(s) + B\overline{y}_2(s)$$

Laplace transforms and derivatives: The crucial feature of the transform from the perspective of ODEs is what it does to derivatives: from the definition, and by integrating by parts, we have

$$\mathcal{L}\{\dot{y}(t)\} = s\overline{y}(s) - y(0) \qquad \& \qquad \mathcal{L}\{\ddot{y}(t)\} = s^2\overline{y}(s) - sy(0) - \dot{y}(0)$$

If we apply the Laplace transform to the ODE, $a\ddot{y} + b\dot{y} + cy = f(t)$, we therefore arrive at the algebraic problem,

$$(as^2 + bs + c)\overline{y} - asy(0) - a\dot{y}(0) - by(0) = \overline{f}(s) \qquad \rightarrow \qquad \overline{y} = \frac{\overline{f}(s) + (as+b)y(0) + a\dot{y}(0)}{as^2 + bs + c},$$

where $\overline{f}(s) = \mathcal{L}\{f(t)\}$. The problem is then broken down into the three steps:

(1) Compute $\overline{f}(s)$ from $f(t)$
(2) Include the initial conditions to calculate $\overline{y}(s)$
(3) Convert $\overline{y}(s)$ back to $y(t)$.

Notes:

- Laplace transform converts an ODE to an algebraic problem for the transform of the unknown function
- There is no need to split the solution into homogeneous and particular pieces
- There is no need to pose any trial solutions
- The initial conditions are automatically incorporated

Inverting the transform: The preceding advantages are, of course, too good to be true: we want to find $y(t)$ not $\overline{y}(s)$. To undo the transform we might try to use the inverse Laplace transform, which is defined as another integral:

$$y(t) = \mathcal{L}^{-1}\{\overline{y}(s)\} = \int_{\mathcal{C}} e^{st}\overline{y}(s)\frac{ds}{2\pi i}.$$

The bad news is that $\mathcal{C}$ is the "Bromwich contour", which is a path over the complex $s-$plane. For Math 256, this definition is useless, as we cannot yet do "path integrals" of this sort. Instead, for step (3), we will build up a repertoir of known Laplace transforms in a table. This table can then be used to recognize what functions of t corresponds to our calculated functions of s, and so we can then write down the desired solution.

e.g. If $\overline{y}(s) = \frac{1}{s}$ then $y(t) = \mathcal{L}^{-1}\{s^{-1}\} = 1$.
If $\overline{y}(s) = \frac{1}{s-m}$ then $y(t) = \mathcal{L}^{-1}\{(s-m)^{-1}\} = e^{mt}$.

Another example of a "transform pair" is $y(t) = t^n$ and $\overline{y}(s) = \frac{n!}{s^{n+1}}$ (Real$(s) > 0$), which can be established by again using the definition of the Laplace transform (and integrating by parts).

Sample ODE problems:

• $\ddot{y} - 2\dot{y} - 3y = 0$ with $y(0) = 0$ and $\dot{y}(0) = 1$. We have

$$(s^2 - 2s - 3)\overline{y} - 1 = 0 \qquad \rightarrow \qquad \overline{y} = \frac{1}{(s+1)(s-3)}$$

The right-hand side is not one of our currently known Laplace transforms. However, a partial fraction comes to the rescue:

$$\frac{1}{(s+1)(s-3)} = \frac{1}{4}\left(\frac{1}{s-3} - \frac{1}{s+1}\right)$$

But $\mathcal{L}\{e^{-t}\} = 1/(s+1)$ and $\mathcal{L}\{e^{3t}\} = 1/(s-3)$ (as long as Real$(s) > 3$). Thus,

$$y(t) = \frac{1}{4}(e^{3t} - e^{-t}).$$

• $\ddot{y} + y = 0$ with $y(0) = 0$ and $\dot{y}(0) = 1$. We have

$$\overline{y} = \frac{1}{s^2+1},$$

which is again not one of our currently known Laplace transforms. This time we need to add more entries to our table. Consider

$$\mathcal{L}\{\sin(\omega t)\} = \int_0^\infty e^{-st}\sin(\omega t)dt = -\left[\frac{e^{-st}}{s}\sin(\omega t)\right]_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos(\omega t)dt.$$

That is, $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s}\mathcal{L}\{\cos(\omega t)\}$ since the first term vanishes on plugging in the limits (provided Real$(s) > 0$). Similarly,

$$\mathcal{L}\{\cos(\omega t)\} = -\left[\frac{e^{-st}}{s}\cos(\omega t)\right]_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin(\omega t)dt = \frac{1}{s} - \frac{\omega}{s}\mathcal{L}\{\sin(\omega t)\}$$

Thus,

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} \qquad \& \qquad \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}.$$

Evidently, for our ODE, $y(t) = \sin t$.

• $\dot{y} + 5y = 2$, $y(0) = 1$. Applying the Laplace transform:

$$\overline{y}(s) = \frac{2+s}{s(s+5)} = \frac{2}{5s} + \frac{3}{5(s+5)} \qquad \rightarrow \qquad y(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

• $\ddot{y} + 4y = 6$, $y(0) = 0$ and $\dot{y}(0) = 5$. Applying the Laplace transform:

$$\overline{y}(s) = \frac{6+5s}{s(s^2+4)} = \frac{3}{2s} - \frac{3}{2}\frac{s}{(s^2+4)} + \frac{5}{2}\frac{2}{(s^2+4)} \qquad \rightarrow \qquad y(t) = \frac{3}{2} - \frac{3}{2}\cos 2t + \frac{5}{2}\sin 2t$$

Helpful inversion tools: the table, partial fractions, shifting theorems (including completing a square)

First shifting theorem: $\mathcal{L}\{e^{at}y(t)\} = \int_0^\infty e^{-(s-a)t}y(t)dt = \overline{y}(s-a)$.

$$y(t) = t \ \rightarrow \ \overline{y}(s) = \mathcal{L}\{t\} = \frac{1}{s^2} \qquad \rightarrow \qquad \mathcal{L}\{te^{at}\} = \overline{y}(s-a) = \frac{1}{(s-a)^2}$$

• $\ddot{y} + \dot{y} - 2y = 9e^t$, $y(0) = 3$ and $\dot{y}(0) = 0$. Applying the Laplace transform:

$$\overline{y}(s) = \frac{3s^2+6}{(s+2)(s-1)^2} = \frac{2}{s+2} + \frac{1}{s-1} + \frac{3}{(s-1)^2}$$

We know the transforms

$$\mathcal{L}\{2e^{-2t}\} = \frac{2}{s+2}, \qquad \mathcal{L}\{e^t\} = \frac{1}{s-1}, \qquad \mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}.$$

Hence $y(t) = 2e^{-2t} + e^t + 3te^t$.

• $\ddot{y} - 4\dot{y} + 13y = 0$, $y(0) = 0$ and $\dot{y}(0) = 3$. Applying the Laplace transform:

$$\overline{y}(s) = \frac{3}{s^2 - 4s + 13} = \frac{3}{(s-2)^2 + 9}$$

We know the transform pair

$$y(t) = \sin 3t, \qquad \overline{y}(s) = \frac{3}{s^2 + 9}$$

and so if we use the first shifting theorem,

$$\overline{y}(s) \equiv \mathcal{L}\{e^{2t} \sin 3t\} \qquad \rightarrow \qquad y(t) = e^{2t} \sin 3t$$

Step functions and the second shifting theorem: the Heaviside step function is defined so that

$$H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

The step function is useful in mathematically describing functions that switch on and off. *e.g.*

$$f(t) = \begin{cases} 0 & t < 0 \\ t(t-1) & 0 < t < 1 \\ 0 & t > 1 \end{cases} \qquad \rightarrow \qquad f(t) = t(t-1)[H(t) - H(t-1)]$$

Now consider

$$\mathcal{L}\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st} f(t-a)H(t-a)dt = e^{-sa} \int_a^\infty e^{-s(t-a)} f(t-a)dt$$

$$= e^{-sa} \int_0^\infty e^{-s\tilde{t}} f(\tilde{t})d\tilde{t} = e^{-sa}\overline{f}(s)$$

which furnishes the second shifting theorem. In other words, an exponential factor in the transofrmed variable corresponds to a time shift.

e.g, $\mathcal{L}^{-1}\{\frac{1}{s}e^{-sa}\} = H(t-a)$ and $\mathcal{L}^{-1}\{\frac{\omega}{s^2+\omega^2}e^{-sa}\} = H(t-a)\sin\omega(t-a)$.

• An ODE with switches: $\ddot{y} + y = H(t) - H(t-1)$ with $y(0) = \dot{y}(0) = 0$. Applying the Laplace transform:

$$\overline{y} = (1 - e^{-s})\left(\frac{1}{s} - \frac{s}{s^2+1}\right) \qquad \rightarrow \qquad y = 1 - \cos t - H(t-1)[1 - \cos(t-1)].$$

The Dirac delta-function: The delta-function has the special property that

$$\int_a^b \delta(t - t_0) f(t)dt = \begin{cases} f(t_0) & \text{provided } a < t_0 < b \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $a > 0$, $\mathcal{L}\{\delta(t-a)\} = e^{-sa}$ and so $\mathcal{L}^{-1}\{e^{-sa}\} = \delta(t-a)$. The delta-function is related to the step function because

$$\int_{-\infty}^t \delta(\tau - t_0)d\tau = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases} \equiv H(t - t_0).$$

i.e. the delta-function is the derivative of the step function.

• Delta functions correspond to impulsive-type forcings of an ODE: consider the oscillator problem,

$$\ddot{y} + \omega^2 y = \delta(t-a), \qquad y(0) = \dot{y}(0) = 0, \quad a > 0.$$

Using the Laplace transform and the second shifting theorem:

$$\overline{y}(s) = \frac{e^{-as}}{s^2+\omega^2} \qquad \to \qquad y(t) = \frac{1}{\omega}H(t-a)\sin\omega(t-a)$$

Thus the oscillator gets kicked into action at $t = a$.

Transfer functions and convolutions: The convolution integral, denoted here by $f * g$, is defined as

$$f * g = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Consider the Laplace transform of this integral:

$$\mathcal{L}\{f * g\} = \int_0^\infty \int_0^t e^{-st} f(t-\tau)g(\tau)d\tau dt$$

By considering the domain of the double integration over the $(t,\tau)-$plane, one can interchange the order of the integrals and then change variables to find that

$$\mathcal{L}\{f * g\} = \int_0^\infty \int_\tau^\infty e^{-st} f(t-\tau)g(\tau)dtd\tau = \int_0^\infty \int_0^\infty e^{-su}e^{-s\tau} f(u)g(\tau)dud\tau = \overline{f}(s)\overline{g}(s)$$

That is, the inverse Laplace transform of a product is a convolution integral.

- Application to ODEs: $\ddot{y} + 4y = g(t)$ with $y(0) = 3$ and $\dot{y}(0) = -1$. We have

$$\overline{y}(s) = \frac{3s - 1 + \overline{g}(s)}{s^2+4}$$

and so

$$y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2t * g(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)\ g(\tau)d\tau.$$

The final term corresponds to the particular solution of the ODE. *i.e.* the forced response of the oscillator. We have successfully written down this solution in terms of an integral for any forcing term $g(t)$. The factor $\mathcal{T}(t-\tau) = \frac{1}{2}\sin 2(t-\tau)$ in the integrand came from the form of the left-hand side of the original ODE; this is the associated "transfer function".

In other words, using the Laplace transform technology, we can write the solution down in terms of a piece that takes care of the initial condition and a convolution of the transfer function and forcing term:

$$y(t) = \{\text{homogeneous solutions accounting for initial conditions}\} + \mathcal{T} * g.$$