Periodic functions and boundary conditions

A function is periodic, with period T, if it repeats itself exactly after an interval of length T. *i.e.* $y(x) = y(x+T)$ for any x. Evidently, the derivatives of $y(x)$ are also periodic (unless the function cannot be differentiated – otherwise just differentiate $y(x) = y(x+T)$).

e.g. $\sin(\omega t)$ and $\cos(\omega t)$ are periodic functions with period $2\pi/\omega$.

ODEs can be solved over an interval of length T subject to periodic boundary conditions, which state that the function and its derivatives at one end of the interval equal their values at the other end of the interval. The solution is then a periodic function with period T.

For example, for a second-order ODE to be solved over the interval $0 < x < T$, two conditions are needed to uniquely specify the solution. The corresponding periodic boundary conditions are $y(0) = y(T)$ and $y'(0) = y'(T)$.

$$e.g. \qquad y'' + y = (1 - 4\pi^2)\cos(2\pi x), \qquad y(0) = y(1),\ y'(0) = y'(1) \quad (\text{so } T = 1).$$

The homogeneous and particular solutions are

$$y_h = A\cos x + B\sin x \qquad \& \qquad y_p = \cos(2\pi x).$$

The general solution is therefore $y = A\cos x + B\sin x + \cos(2\pi x)$. Applying the boundary conditions: $A + 1 = A\cos 1 + B\sin 1 + 1$ and $B = B\cos 1 - A\sin 1$, which can be solved to find that $A = B = 0$. Thus $y = \cos(2\pi x)$. This solution can be found more straightforwardly by recognizing that $\cos(2\pi x)$ is periodic with period $T = 1$, as needed, but the two homogeneous solutions $\cos x$ and $\sin x$, though periodic, have the wrong period (2π) and must therefore be eliminated.

Fourier series

A periodic function $f(x)$ with period $T = 2\pi$ can be represented by a Fourier series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \qquad (*)$$

for some constant a_0 and a set of (constant) coefficients $\{a_n, b_n\}$, $n = 1, 2, \ldots$

Helpful integrals: for any integers n and m,

$$\int_{-\pi}^{\pi}\cos nx\ dx = \int_{-\pi}^{\pi}\sin nx\ dx = \int_{-\pi}^{\pi}\cos nx\sin mx\ dx = 0$$

$$\int_{-\pi}^{\pi}\cos nx\cos mx\ dx = \int_{-\pi}^{\pi}\sin nx\sin mx\ dx = \begin{cases} 0, & \text{if } n \neq m \\ \pi, & \text{if } n = m \end{cases}$$

which follow on using our handy trig formulae.

Using the preceding integrals, one can determine the constant and coefficients of the Fourier series by multiplying $(*)$ by one of 1, $\cos mx$ or $\sin mx$, and then integrating x from $-\pi$ to π. We obtain

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\ dx, \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\ dx$$

e.g. The sawtooth function, $f(x) = \frac{\pi}{2} - |x|$ for $-\pi < x < \pi$, with $f(x) = f(x+2\pi) = f(x-2\pi)$ furnishing the function for points outside this interval. We find

$$\pi a_0 = \int_0^{\pi}\left(\frac{\pi}{2} - x\right)dx + \int_{-\pi}^{0}\left(\frac{\pi}{2} + x\right)dx = 0,$$

$$\pi a_m = \int_0^{\pi}\left(\frac{\pi}{2} - x\right)\cos mx\ dx + \int_{-\pi}^{0}\left(\frac{\pi}{2} + x\right)\cos mx\ dx = \frac{2}{m^2}[1 - (-1)^m] \qquad (\text{integrating by parts})$$

and

$$\pi b_m = \int_0^\pi \left(\frac{\pi}{2} - x\right) \sin mx \, dx + \int_{-\pi}^0 \left(\frac{\pi}{2} + x\right) \sin mx \, dx = 0.$$

Hence

$$\frac{\pi}{2} - |x| = \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[1 - (-1)^n] \cos nx = \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos[(2j-1)x]}{(2j-1)^2} = \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right).$$

(given $a_n = 0$ for n even, and then putting $n = 2j - 1$ for n odd).

Application to an ODE: If $f(x)$ is the preceding sawtooth, solve

$$y'' - y = -f(x), \qquad y(-\pi) = y(\pi) \ \& \ y'(-\pi) = y'(\pi).$$

We use the fact that $f(x)$ can be represented as the Fourier series above (with $a_0 = b_n = 0$). Hence,

$$y'' - y = -\sum_{n=1}^{\infty} a_n \cos nx, \qquad a_n = \frac{2}{n^2\pi}[1 - (-1)^n].$$

The homogeneous solutions are $y_h = Ae^x + Be^{-x}$. The inhomogeneous term is a series of cosines, so we pose the trial particular solution,

$$y_p = \sum_{n=1}^{\infty} d_n \cos nx.$$

Plugging nto the ODE gives

$$\sum_{n=1}^{\infty}(1 + n^2)d_n \cos nx = \sum_{n=1}^{\infty} a_n \cos nx,$$

then matching up each of the cosines gives $d_n = a_n/(1 + n^2)$. The general solution is therefore

$$y = Ae^x + Be^{-x} + \sum_{n=1}^{\infty} \frac{a_n \cos nx}{n^2 + 1}.$$

However, the periodic boundary conditions demand that $y(x)$ be periodic with period 2π, whereas the homogeneous solution is never periodic, and so $A = B = 0$. Alternatively, one can substitute the general solution into the boundary conditions, and solve the resulting pair of algebraic equations to find these values of A and B explicitly. Thus,

$$y = \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos[(2j-1)x]}{[(2j-1)^2 + 1](2j-1)^2}.$$

The square wave: Find the Fourier series for

$$f(x) = \begin{cases} k & \text{for } x > 0 \\ -k & \text{for } x < 0 \end{cases}, \qquad f(x) = f(x + 2\pi).$$

We find

$$\pi a_0 = k\int_0^\pi dx - k\int_{-\pi}^0 dx = 0, \qquad \pi a_n = k\int_0^\pi \cos nx \, dx - k\int_{-\pi}^0 \cos nx \, dx = 0$$

and

$$\pi b_n = k\int_0^\pi \sin nx \, dx - k\int_{-\pi}^0 \sin nx \, dx = \frac{2k}{n}[1 - (-1)^n].$$

Hence (given $b_n = 0$ for n even, and then setting $n = 2j - 1$ for n odd),

$$f(x) = \frac{4k}{\pi} \sum_{j=1}^{\infty} \frac{\sin[(2j-1)x]}{(2j-1)} = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right).$$

Note that the sawtooth is an even function (*i.e.* $f(x) = f(-x)$) and has $b_n = 0$, whereas the square wave is an odd function (*i.e.* $f(x) = -f(-x)$) and has $a_0 = a_n = 0$. These are, in fact, general properties of even and odd functions (which reduces by at least one half the degree of effort required to compute their Fourier series!).

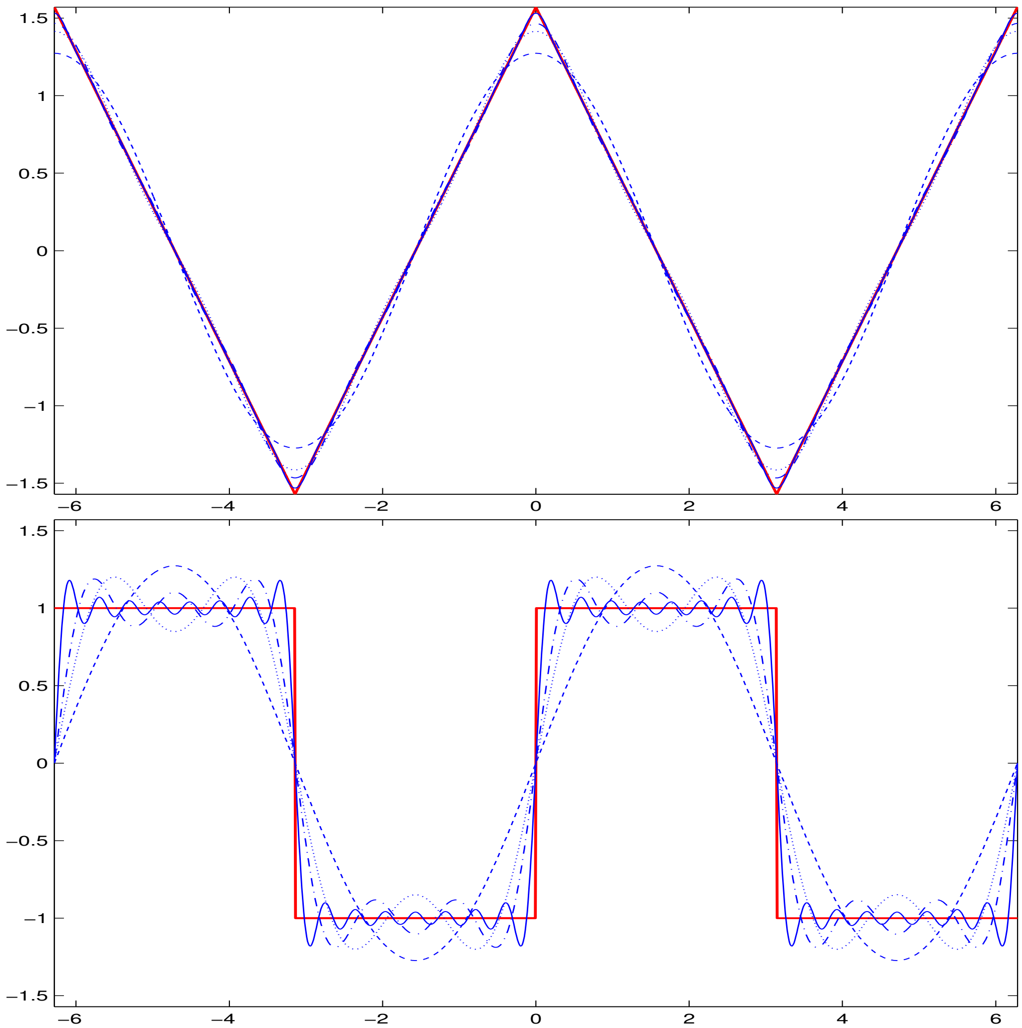

Figure 1: The sawtooth and square wave. The functions are shown in (thick) red. The blue lines show the Fourier series truncated at $j = 1$ (dashed), 2 (dotted), 3 (dash-dotted) and 8 (solid). Note the Gibbs phenomenon for the square wave (residual oscillations in the truncated Fourier series).

Funks with Jumps

If $f(x)$ has a discontinuity at $x = a$, the Fourier series converges to $\frac{1}{2}f(a^-) + \frac{1}{2}f(a^+)$, where $f(a^-)$ is the limit of $f(x)$ as x approaches a from the left, and $f(a^+)$ is the limit of $f(x)$ as x approaches a from the right. Thus, if $f(x)$ is defined to be anything other than $\frac{1}{2}f(a^-) + \frac{1}{2}f(a^+)$ at the jump, the Fourier series will not converge to $f(x)$ at $x = a$.

At any jumps of $f(x)$, the Fourier series displays persistent "ringing" if truncated at a finite number of terms because smooth functions (*i.e.* sines and cosines) are being used to represent something that is discontinuous. This is "Gibbs phenomenon".

Even and odd functions

A function is *even* if $f(x) = f(-x)$; it is *odd* if $f(x) = -f(-x)$. In view of these properties

$$\int_{-\pi}^{\pi} (\text{Even Function})dx = 2\int_{0}^{\pi} (\text{Even Function})dx \qquad \& \qquad \int_{-\pi}^{\pi} (\text{Odd Function})dx = 0.$$

The cosine function is even ($\cos(-x) = \cos x$), and the sine function is odd ($-\sin(-x) = \sin x$).

• Products of even function remain even; *i.e.* if $f(x)$ and $g(x)$ are both even, then $f(x)g(x)$ is even.
• Products of odd function are also even: if $f(x)$ and $g(x)$ are both odd, then $f(x)g(x)$ is even.
• The product of an even and an odd function is odd: if $f(x)$ is even and $g(x)$ is odd, then $f(x)g(x)$ is odd.
All of these follow from switching the sign of x in the arguments and then using the properties of f and g.

Thus,

$$\int_{-\pi}^{\pi}(\text{Even Function})\cos nx\,dx = 2\int_{0}^{\pi}(\text{Even Function})\cos nx\,dx \qquad \& \qquad \int_{-\pi}^{\pi}(\text{Even Function})\sin nx\,dx = 0,$$

and

$$\int_{-\pi}^{\pi}(\text{Odd Function})\cos nx\,dx = 0 \qquad \& \qquad \int_{-\pi}^{\pi}(\text{Odd Function})\sin nx\,dx = 2\int_{0}^{\pi}(\text{Odd Function})\sin nx\,dx.$$

This means that for an even function, the coefficients of the sine terms of the Fourier series must vanish ($b_n = 0$), and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{with} \qquad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \quad \& \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

This is called a "Fourier cosine series".

Similarly, for an odd function, the constant and the coefficients of the cosine terms of the Fourier series must vanish ($a_0 = a_n = 0$), and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{with} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

This is called a "Fourier sine series".

Fourier series for arbitrary period

A periodic function with period $T = 2L$ can be represented by the Fourier series,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx;$$

the proof follows as before, but using the alternative handy integrals

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\,dx = \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\,dx = \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = 0$$

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx = \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} 0, & \text{if } n \neq m \\ L, & \text{if } n = m \end{cases}$$

Again, even functions have the Fourier cosine series,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_0 = \frac{2}{L}\int_0^{L} f(x)dx, \quad a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx,$$

and odd functions have the Fourier sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

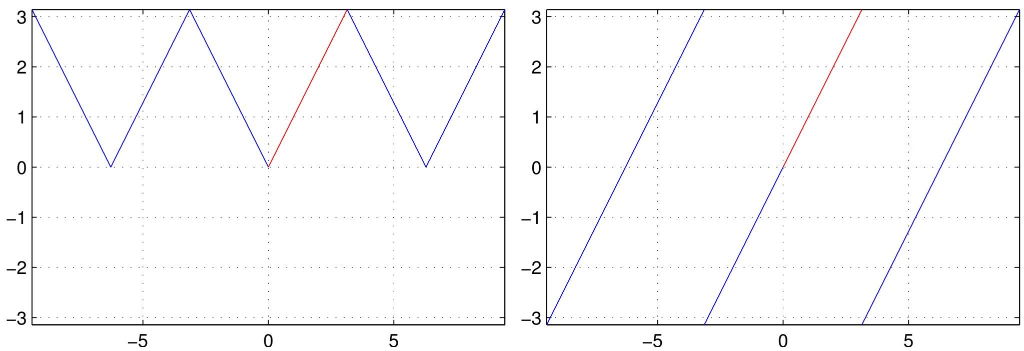

Figure 2: The even and odd periodic extensions of the function $f(x) = x$ for $0 < x < \pi$.

Extensions of functions

If a function $f(x)$ is defined over the interval $0 < x < L$, then it can be extended to $-L < x < L$ as an even function by defining $f(x) = f(-x)$ for $x < 0$. Similarly, the function can be extended to $-L < x < L$ as an odd function by demanding that $f(-x) = -f(x)$ for $x < 0$.

Both of these new functions can then be made periodic with period $2L$ by also demanding that $f(x) = f(x+2L) = f(x-2L)$ if x lies outside of the interval $(-L, L)$. The first is the "even periodic extension" of $f(x)$ and is described by a Fourier cosine series. The second is the "odd periodic extension" of $f(x)$ and has a Fourier sine series.

The odd periodic extension of $f(x)$ necessarily satisfies $f(0^-) + f(0^+) = 0$ and $f(L^-) + f(-L^+) = 0$. In view of the periodicity condition $f(x) = f(x \pm 2L)$, the latter further implies that $f(L^-) + f(L^+) = 0$. The Fourier series respresentation of the odd periodic extension of the function $f(x)$ will therefore vanish at $x = 0$ and L. These ideas of extension are handy results when we start solving PDEs, allowing us to use Fourier series theory to compute the coefficients of general solutions.