PDEs

For a function, $u(x,t)$, of two variables, x and t, the general form of a PDE is

$$F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt},...)=0$$

for some function F, where

$$u_x=\frac{\partial u}{\partial x},\quad u_t=\frac{\partial u}{\partial t},\quad u_{xx}=\frac{\partial^2 u}{\partial x^2},\quad u_{tt}=\frac{\partial^2 u}{\partial t^2},\quad u_{xt}=\frac{\partial^2 u}{\partial x\partial t},\quad etc.$$

i.e. subscripts denote shorthand notation for partial derivatives.

Three important PDEs:

$$\text{Heat or diffusion equation :}\quad u_t=\kappa u_{xx}$$

$$\text{Wave equation :}\quad u_{tt}=c^2u_{xx}$$

$$\text{Laplace's equation :}\quad u_{xx}+u_{yy}=0$$

For the heat equation, κ is the "diffusivity", and in the wave equation we see the "wavespeed" c (in this course, we will mostly scale variables so that these dimensional constants can be taken to be unity).

Initial and boundary conditions: As for ODEs, PDEs must be solved subject to initial/boundary conditions in order to uniquely specify the solution. For a PDE, these conditions must be stated for both variables. The maximum number of derivatives for each of the variables indicates how many initial/boundary conditions are needed.

e.g. Heat equation (one $t-$derivative and two $x-$derivatives), $u_t=\kappa u_{xx}$ with $u(x,0)=f(x)$ (a specified initial function of x) and $u(0,t)=u(L,t)=0$. In the problem of heat conduction, this would model how the temperature evolves along a rod of length L whose ends are maintained at zero degrees and in which the temperature is initially given by $f(x)$.

Sample solutions:

• The heat equation $u_t=u_{xx}$, with $u(0,t)=u(\pi,t)=0$ and $u(x,0)=\sin x$, has solution $u(x,t)=e^{-t}\sin x$ (as can be verified by substitution of the solution into the PDE).

• The wave equation $u_{tt}=u_{xx}$, with $u(0,t)=u(\pi,t)=0$, $u(x,0)=0$ and $u_t(x,0)=\sin x$ (there are two initial/boundary conditions in both x and t because there are two time and space derivatives), has solution $u(x,t)=\sin x\sin t$.

The wave equation $u_{tt}=c^2u_{xx}$ also has a general solution, called d'Alembert's solution, $u(x,t)=F(x-ct)+G(x+ct)$ for two arbitrary functions $F(z)$ and $G(z)$ (this follows because $u_{xx}=F''+G''$ and $u_{tt}=c^2(F''+G'')$, on using the chain rule). $F(x-ct)$ represents a disturbance that propagates to the right at speed c without any change in shape; $G(x+ct)$ is a disturbance that steadily propagates to the left.

Separation of Variables

Our solution strategy for solving PDEs is to turn them into a set of ODEs. Let's proceed using the heat equation as an example:

$$u_t=u_{xx},\quad u(0,t)=u(\pi,t)=0,\quad u(x,0)=f(x),$$

for some prescribed $f(x)$. Let $u(x,t)=X(x)T(t)$; *i.e.* we separate the variables. Then

$$u_t=XT'\qquad \&\qquad u_{xx}=TX'',$$

where the prime indicates the derivative with respect to the argument. Plugging into the PDE and dividing by XT gives $T'/T=X''/X$. But the left-hand side is a function only of t, whereas the right-hand side is only a function of x. This can only be true if both equal a constant! Thus

$$\frac{T'}{T}=\frac{X''}{X}=-\lambda$$

where λ is the "separation constant". Now we have the two ODEs,

$$T' = -\lambda T \qquad \& \qquad X'' = -\lambda X.$$

Thus,

$$T = Ce^{-\lambda t} \qquad \& \qquad X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

So far, we have not figured out what λ should be in order to solve the problem. At our disposal for this task are the boundary conditions in x: $u(0,t) = u(\pi,t) = 0$, which imply $X(0) = X(\pi) = 0$. Using the solution for $X(x)$, we find that

$$A = 0 \qquad \& \qquad B\sin(\sqrt{\lambda}\pi) = 0.$$

We could take $B = 0$, but that gives the trivial solution $u = 0$ which cannot satisfy the initial condition $u(x,0) = f(x)$ for a finite $f(x)$. The other way to solve the second boundary condition is to take $\sqrt{\lambda} = n$ for $n = 1, 2, ...$ That is,

$$\lambda = n^2, \qquad X = B\sin nx \qquad \& \qquad T = Ce^{-n^2 t}.$$

At this stage, given certain initial functions $f(x)$ it is clear how we solve the PDE. *e.g.* for $u(x,0) = \sin x$, we should clearly take $n = 1$, $B = C = 1$ to furnish the solution $u(x,t) = e^{-t}\sin x$ quoted earlier.

More generally, we realize that there is an infinite set of possible solutions corresponding to all the integer values for n. Hence we may write a general solution by combining all of them into a sum:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx).$$

Note that the solution for X and that for T both include an arbitrary constant, so when we combine them back into u, we have the arbitrary combination BC, which can be written as another (but single) arbitrary constant. Moreover, when we combine all the possible solutions into the general sum for $u(x,t)$, we should take a different arbitrary factor for each n to remain as general as possible. Hence, we made the replacement $BC \to b_n$ in the sum above.

To make the general solution satisfy the initial condition, we must finally demand that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

In other words, we need to represent $f(x)$ as a Fourier sine series! This is possible because both $f(x)$ and $u(x,t)$ are only currently defined on the interval $0 < x < \pi$. We may then make the **odd periodic** extension of both to define them beyond that interval. Moreover, by taking the odd extension we ensure that $f(x)$ can be represented by a Fourier sine series, as is required. The coefficients b_n are therefore set by

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx,$$

and we have solved the PDE for any old $f(x)$!

e.g. $f(x) = \sin Nx$ for some integer N. In view of the helpful integrals quoted in the Fourier series notes, we find $b_n = 0$ for $n \neq N$ and $b_N = 1$. Hence $u(x,t) = e^{-N^2 t}\sin Nx$.

e.g. $f(x) = x(\pi - x)$. We just need to compute

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\, dx = \frac{4}{\pi n^3}\left[1 - (-1)^n\right]$$

(by two integrations by parts), and then plug this result into our general solution.

A second example, using Laplace's equation:

$$u_{xx} = u_{yy}, \qquad u_x(0,y) = u_x(L,y) = 0, \qquad u(x,0) = f(x) \qquad \& \qquad u_y \to 0 \text{ for } y \to \infty.$$

We separate variables: $u = X(x)Y(y)$ and divide by XY, giving

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda,$$

where λ is another separation constant. Hence,

$$X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \qquad \& \qquad Y = Ce^{-\sqrt{\lambda}y} + De^{-\sqrt{\lambda}y}$$

for four constants, A, B, C and D. The boundary conditions in x imply $X'(0) = X'(L) = 0$. Hence

$$B = 0 \qquad \& \qquad A\sqrt{\lambda}\sin\sqrt{\lambda}L = 0.$$

Thus, avoiding a trivial solution, we need to take $\lambda = n^2\pi^2/L^2$, with $n = 1, 2, ...$ This time, however, $\lambda = 0$ and $Y = \textit{constant}$ is also a (non-trivial) solution. Hence,

$$X = A\cos\left(\frac{n\pi x}{L}\right) \qquad \& \qquad Y = Ce^{-n\pi y/L} + De^{n\pi y/L} \qquad (\lambda = n^2\pi^2/L^2),$$

or $\lambda = 0$ implying both X and Y are constants. Thus, we end up with a general solution for $u(x, y)$ that we write in the form

$$u = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n e^{-n\pi y/L} + \hat{a}_n e^{n\pi y/L})\cos\left(\frac{n\pi x}{L}\right)$$

(minimizing the needed number of arbitrary constants to a_0, a_n and $\hat{a}_n$).

Now, the boundary condition $u_y \to 0$ for $y \to \infty$ is impossible to solve unless $\hat{a}_n = 0$. With this choice, $u(x, 0) = f(x)$ implies that

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

That is, $f(x)$ must be expressed as a Fourier cosine series. To accomplish this task, we make the **even periodic** extension of $f(x)$ beyond $0 < x < L$, implying that

$$a_0 = \frac{2}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx,$$

and feed the result into the general solution.

General tips:

• It is helpful to make a good choice for the sign of the separation constant. Usually, when there are zero BCs on u or its derivative that are given for one of the spatial variables, a good choice is to make sure that one obtains trig functions when one solves for the corresponding part of u. *i.e.* in the two problems above, zero BCs in x were provided on u or u_x. The good choice for the sign of the separation constant was therefore the one that gave $X'' + \lambda X = 0$. If one puts in the opposite sign (*i.e.* to obtain $X'' - \lambda X = 0$), one instead finds that the solutions in x are exponential and it is then not possible to satisfy both BCs (unless one takes λ to be negative, switching the exponentials back into trig funks).

• Usually, one is given some simple BCs in one of the variables, and a more complicated BC in the other variable in which one must recover a prescribed function. In the separation of variable method, one fixes λ using the (trig) functions satisfying the simple boundary conditions, and then assembles the full general solution (a Fourier series for Math 256) before trying to recover the prescribed function in the more complicated boundary condition. That final step requires suitably extending the prescribed function beyond the domain in which one is solving the PDE and using results from Fourier series theory to calculate the unknown coefficients of the general solution.

• $u = 0$ at $x = 0$ and L usually give a Fourier sine series for the solution in x; $u_x = 0$ at $x = 0$ and L usually give a Fourier cosine series. Other cases are possible too, given a mix of such BCs.

Inhomogeneous boundary conditions and steady states for the heat equation

For heat flow in a rod, a boundary condition of the form $u = a$ corresponds to holding that end at fixed temperature; a boundary condition of the form $u_x = b$ corresponds to imposing the flux of heat. But if a and b here are not zero (*i.e.* the boundary conditions are inhomogeneous), the separation of variables method outlined above fails.

e.g. $u(0,t) = c_1$ and $u(L,t) = c_2$. In this case, separation of variables gives $X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$ with $A = c_1$ and $A\cos\sqrt{\lambda}L + B\sin\sqrt{\lambda}L = c_2$, which does not generate the infinite set of $\lambda-$values for our Fourier series.

To save the situation, we use the *steady-state solution* to the PDE. This is time-independent and statisfies the ODE

$$\frac{\partial^2 u_{ss}}{\partial x^2} = 0 \qquad \longrightarrow \qquad u_{ss}(x) = \alpha x + \beta$$

for two arbitrary constants α and β. If the boundary conditions are $u_{ss}(0) = c_1$ and $u_{ss}(L) = c_2$, then $u_{ss}(x) = c_1 + (c_2 - c_1)\frac{x}{L}$.

Returning to the PDE, we now set

$$u(x,t) = v(x,t) + u_{ss}(x).$$

Because $u_{ss}(x)$ already takes care of the inhomogeneous boundary conditions on $u(x,t)$, the new function $v(x,t)$ must now be taken to satisfy the homogeneous boundary conditions, $v(0,t) = v(L,t) = 0$. Moreover, $u_t = v_t$ and $u_{xx} = v_{xx}$, and so $v(x,t)$ also solves the heat equation. But because the boundary conditions on $v(x,t)$ are homogeneous, the separation of variable method now works fine to compute the solution.

e.g.

$$u_t = u_{xx}, \qquad u(0,t) = c_1, \quad u(L,t) = c_2, \quad u(x,0) = f(x).$$

The steady-state solution is $u_{ss}(x) = c_1 + (c_2 - c_1)\frac{x}{L}$. Hence, we set $u(x,t) = c_1 + (c_2 - c_1)\frac{x}{L} + v(x,t)$. The new function v satisfies the problem

$$v_t = v_{xx}, \qquad v(0,t) = 0, \quad v(L,t) = 0, \quad v(x,0) = f(x) - c_1 - (c_2 - c_1)\frac{x}{L}.$$

Separation of variables now gives

$$v = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right),$$

with

$$b_n = \frac{2}{L}\int_0^L \left[f(x) - c_1 - (c_2 - c_1)\frac{x}{L}\right] \sin\left(\frac{n\pi x}{L}\right) dx.$$

e.g.

$$u_t = u_{xx}, \quad u(0,t) = 0, \quad u_x(1,t) = 1 \quad \& \quad u(x,0) = f(x).$$

The BCs are inhomogeneous, so we first find the steady-state solution, which is $u_{ss}(x) = x$. Then we set $u(x,t) = x + v(x,t)$ and solve

$$v_t = v_{xx}, \quad v(0,t) = 0, \quad v_x(1,t) = 0 \quad \& \quad v(x,0) = f(x) - x.$$

Separation of variables with $v(x,t) = X(x)T(t)$ now gives

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda \qquad \longrightarrow \qquad X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \quad \& \quad T = Ce^{-\lambda t}.$$

The boundary conditions in x imply

$$A = 0 \qquad \& \qquad B\sqrt{\lambda}\cos\sqrt{\lambda} = 0.$$

Non-trivial solutions are avoided by taking

$$\sqrt{\lambda} = (m - \tfrac{1}{2})\pi, \qquad X = B\sin[\pi(m - \tfrac{1}{2})x] \qquad \& \qquad T = Ce^{-\pi^2(m-\frac{1}{2})^2 t}, \qquad m = 1, 2, ...$$

We then formulate the general solution

$$v(x,t) = \sum_{m=1}^{\infty} B_m e^{-\pi^2(m-\frac{1}{2})^2 t} \sin[\pi(m - \tfrac{1}{2})x].$$

Unfortunately, a fly is now stuck in our Fourier-series ointment: this does not look like a Fourier sine series because $m - \frac{1}{2}$ appears instead of an integer inside the sine functions. However, for a period of $2L = 4$, an odd periodic function has the sine series

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right), \quad b_n = \int_0^2 F(x)\sin\left(\frac{n\pi x}{2}\right)\, dx.$$

Therefore, if $b_n = 0$ for n even and we put $n = 2m - 1$ and $b_n = B_m$ otherwise, then the Fourier sine series does correspond to what we have in our general solution for $v(x,t)$. In other words, the general solution is a Fourier series, but for an odd function that is extended from $0 < x < 1$ to $0 < x < 2$ in such a way that $b_n = 0$ for even n. Regardless of this, we can calculate our B_m coefficients by multiplying $v(x,0)$ by $\sin\pi(M - \frac{1}{2})x$ for a particular integer M and then integrating over x using the handy result

$$\int_0^1 \sin[\pi(m - \tfrac{1}{2})x]\sin[\pi(M - \tfrac{1}{2})x]\, dx = \begin{cases} 0 & m \neq M \\ \frac{1}{2} & m = M \end{cases}$$

which gives

$$B_M = 2\int_0^1 [f(x) - x]\sin[\pi(M - \tfrac{1}{2})x]\, dx.$$

An inhomogeneous PDE

The trick of using the steady-stare solution also works if the PDE includes an inhomogeneous term. *e.g.*

$$u_t = u_{xx} + 1, \quad u(0,t) = 0, \quad u(1,t) = 0 \quad \& \quad u(x,0) = f(x).$$

The steady-state solution $u_{ss}(x)$ satisfies

$$u_{ss}'' = -1, \quad u_{ss}(0) = u_{ss}(1) = 0,$$

with solution $u_{ss}(x) = \frac{1}{2}x(1-x)$.

We next set $u(x,t) = \frac{1}{2}x(1-x) + v(x,t)$ and solve the problem

$$v_t = v_{xx}, \quad v(0,t) = 0, \quad v(1,t) = 0 \quad \& \quad v(x,0) = f(x) - \tfrac{1}{2}x(1-x).$$

The solution (by separation of variables and Fourier-series theory) is

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin(n\pi x), \qquad b_n = 2\int_0^1 [f(x) - \tfrac{1}{2}x(1-x)]\sin(n\pi x)dx.$$

e.g. if $f(x) = 0$, we integrate by parts twice to find that

$$b_n = -\frac{2}{n^3\pi^3}\left[1 - (-1)^n\right].$$

Note that the last calculation implies that

$$\tfrac{1}{2}x(1-x) = \sum_{m=1}^{\infty} \frac{4\sin[(2m-1)\pi x]}{\pi^3(2m-1)^3}.$$

Taking $x = \frac{1}{2}$ gives

$$\frac{1}{8} = \sum_{m=1}^{\infty} \frac{4(-1)^{m+1}}{\pi^3(2m-1)^3}.$$

In other words, Fourier series can be used to sum certain series!