Math 257/316 Assignment 1

Due Wednesday September 10 IN CLASS

Problem 1: (**ODE Review**)Find the general solutions of the following equations:

a. $xy' + 3y = xe^x$, Exact solution is: $y(x) = e^x - \frac{3}{x}e^x + \frac{6}{x^2}e^x - \frac{6}{x^3}e^x + \frac{1}{x^3}C_1$

b. $y' = e^x/y^2$, Exact solution is: $y(x) = \sqrt[3]{(3e^x + C_1)}$

c. $y'' + 2y' + y = 0$, Exact solution is: $y(x) = C_1e^{-x} + C_2e^{-x}x$

d. $y'' - 2y' + 5y = 0$, Exact solution is: $y(x) = C_1e^x \sin 2x + C_2e^x \cos 2x$

e. $\begin{array}{l} 2y'' - 3y' - 2y = 0 \\ y(0) = -1 \\ y'(0) = 3 \end{array}$, Exact solution is: $y(x) = e^{2x} - 2e^{-\frac{1}{2}x}$

f. $3x^2y'' - xy' + y = -x^2 - x^{4/3}$, Exact solution is: $y(x) = -\frac{1}{5}x^2 - x^{\frac{4}{3}} + C_1x + C_2\sqrt[3]{x}$

g. $4x^2y'' + y = 0$, Exact solution is: $y(x) = C_1\sqrt{x} + C_2\sqrt{x}\ln x$

h. $x^2y'' + 3xy' + 2y = 0$, Exact solution is: $y(x) = \frac{C_1}{x}\sin(\ln x) + \frac{C_2}{x}\cos(\ln x)$

Problem 2: (**Power series solution warm-up**): Consider the following first order linear ODEs:

$$(1 + x)y' + y = 0 \quad \#$$

$$y' + (1 + x)y = 0 \quad \#$$

a. Solve the differential equations () and () using the appropriate integrating factors.

b. Expand these solutions in Taylor series about the point $x_0 = 0.$ For what values of x do these series fail to converge?

c. Now assume a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \#$$

For each of the differentail equations () and () assume a series solution of the form () and obtain a recursion for the coefficients a_n. Use these recursions to determine the series representations of the solutions. Compare this result to the series obtained in part b above. For equation () is there any relationship between the points of divergence of the series and the coefficient $(1 + x)$ of the derivative in ()?

d. Consider the following recursive strategy to generate an approximate solution to (). Rewrite () as

$$y' + y = -xy \quad \#$$

Now assuming $x \to 0$ and discarding the right hand side of (), find a first order approximation y_0 as the solution to

$$y_0' + y_0 = 0$$

Now substitute y_0 on the right side of () and solve for y_1

$$y_1' + y_1 = -xy_0$$

Continue this process till you obtain $y_2.$ How does y_2 compare with the solutions obtained in a and c?

2)

$$(1+x)y' + y = 0 \qquad (1)$$
$$y' + (1+x)y = 0 \qquad (2)$$

a) (1) $\int \frac{dy}{y} = -\int \frac{dx}{(1+x)} + c \Rightarrow \ln y = \ln(1+x)^{-1} + c \Rightarrow y = \frac{A}{1+x}$

OR $[(1+x)y]' = (1+x)y' + y = 0 \quad \therefore \; y = \frac{A}{1+x}$

(2) $\int \frac{dy}{y} = -\int (1+x)dx + c \Rightarrow \ln y = -x - \frac{x^2}{2} + c \quad y = Ae^{-x-\frac{x^2}{2}}$

OR $F = e^{\int 1+x\,dx} = e^{x+x^2/2}$

$\therefore \; [e^{(x+x^2/2)}y]' = e^{x+x^2/2}y' + e^{x+x^2/2}(1+x)y = 0$

$\therefore \; y = Ae^{-x-x^2/2}$

b) (1) $y = \frac{A}{1+x} = A[1-x+x^2-\ldots]$ A GEOMETRIC SERIES

WHICH CONVERGES PROVIDED $|x|<1$.

(2) $y = Ae^{-x}e^{-x^2/2}$

$= A[1-x+x^2/2-\cdots][1-x^2/2+x^4/8-\cdots]$ CONVERGE FOR $|x|<\infty$

$= A[1-x+x^3/3-x^4/12+\cdots]$

c) 1) $Ly = (1+x)y' + y = 0 \qquad (1)$

$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$Ly = y' + xy' + y = 0$

$= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$\boxed{\begin{array}{l} m = n-1 \\ n = m+1 \\ n=1 \Rightarrow m=0 \end{array}} \qquad \boxed{m=n} \qquad \boxed{m=n}$

$= \sum_{m=0}^{\infty} (m+1)a_{m+1}x^m + \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m = 0$

$= [1.a_1 + a_0]x^0 + \sum_{m=1}^{\infty} [(m+1)a_{m+1} + (m+1)a_m]x^m = 0$

$x^0]$ $\quad a_1 = -a_0$

$x^m, m\geq 1]$ $\quad a_{m+1} = -a_m$

$\therefore \; a_2 = -a_1 = a_0 \quad a_3 = -a_2 = -a_0 \quad \ldots$

$y(x) = a_0[1-x+x^2-x^3+\ldots] = a_0/(1+x) \qquad |x|<1$

NOTE: THE SERIES DIVERGES WHEN $x=-1$ AT WHICH POINT THE COEFFICIENT $(1+x)$ OF y' IN (1) VANISHES.

c) CONTINUED (2) $Ly = y' + (1+x)y = 0$ (2)

$$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$Ly = y' + y + xy = 0$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$\boxed{\begin{array}{l} m=n-1 \\ n=m+1 \\ n=1 \Rightarrow m=0 \end{array}} \quad \boxed{m=n} \quad \boxed{\begin{array}{l} m=n+1 \\ n=m-1 \\ n=0 \Rightarrow m=1 \end{array}}$$

$$= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=0}^{\infty} a_m x^m + \sum_{m=1}^{\infty} a_{m-1} x^m$$

$$= [a_1 + a_0] x^0 + \sum_{m=1}^{\infty} \left[(m+1) a_{m+1} + a_m + a_{m-1}\right] x^m$$

$x^0]$ $\quad a_1 = -a_0$

$x^m, m \geq 1]$ $\quad a_{m+1} = \dfrac{-a_m - a_{m-1}}{m+1}$

$$a_2 = \frac{-a_1 - a_0}{2} = \frac{+a_0 - a_0}{2} = 0$$

$$a_3 = \frac{-a_2 - a_1}{3} = +a_0/3$$

$$a_4 = \frac{-a_3 - a_2}{4} = -a_0/12$$

$$\therefore \quad y(x) = a_0 [1 - x + x^3/3 - x^4/12 + \cdots]$$

d) $\quad Ly = y' + (1+x)y = 0$ (2)

$$L_0 y_0 = y_0' + y_0 = 0 \Rightarrow y_0 = c_0 e^{-x}$$

$$L_0 y_1 = y_1' + y_1 = -x(c_0 e^{-x})$$

$$\therefore \quad [e^x y_1]' = -c_0 x \Rightarrow y_1 = -c_0 \frac{x^2}{2} e^{-x} + c_1 e^{-x}$$

$$L_0 y_2 = y_2' + y_2 = -x y_1 = -x\left[-c_0 \frac{x^2}{2} e^{-x} + c_1 e^{-x}\right]$$

$$[e^x y_2]' = +c_0 \frac{x^3}{2} - c_1 x$$

$$\therefore \quad y_2 = \left[c_0 \frac{x^4}{8} - c_1 \frac{x^2}{2} + c_2\right] e^{-x}$$

SINCE A 1ST ORDER EQ CAN ONLY HAVE ONE FREE CONSTANT WE REQUIRE THAT $C_0 = C_1 = C_2$ SO THAT

$$y_2(x) = C_0 e^{-x} \left[1 - \frac{x^2}{2} + \frac{1}{2}\left(\frac{x^2}{2}\right)^2\right]$$

WHICH ARE THE FIRST THREE TERMS IN THE TAYLOR EXPANSION OF $e^{-x^2/2}$ IN THE SOLUTION $\quad y(x) = C_0 e^{-x} e^{-x^2/2}$