Actual Midterm exam - 2018

Closed book exam; no calculators. Adequately explain the steps you take.

1. Use separation of variables to solve

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

outside the unit disk, $r \geq 1$, subject to

$$u(1,\theta) = \begin{cases} 0 & \theta = 0 \text{ or } \pi \\ \pi/2 & 0 < \theta < \pi \\ -\pi/2 & \pi < \theta < 2\pi \end{cases}.$$

Using

$$\frac{1}{2}\ln\left(\frac{1+\psi}{1-\psi}\right) = \sum_{n>0, n\ odd} \frac{\psi^n}{n},$$

sum the series for $u(r,\theta)$, and hence write down a compact logarithmic expression for the solution.

2. Given that $J_1(z) = -J_0'(z)$, use Bessel's equation with $m = 0$ to establish that

$$zJ_1(z) = \int_0^z \hat{z}J_0(\hat{z})d\hat{z}, \qquad \frac{d}{dz}\left[z^2(J_1)^2\right] + z^2\frac{d}{dz}(J_0)^2 = 0, \qquad \int_0^z \hat{z}[J_0(\hat{z})]^2 d\hat{z} = \frac{1}{2}z^2\left[(J_1)^2 + (J_0)^2\right].$$

Use separation of variables to solve

$$u_{tt} = \frac{1}{r}(ru_r)_r$$

inside the unit disk $r \leq 1$, subject to $u(1,t) = 0$, $u(r,0) = 0$ and $u_t(r,0) = 1$. Express your result as a sum involving Bessel functions without any integrals.

Helpful information:

Bessel's equation is

$$z^2y'' + zy' + (z^2 - m^2)y = 0,$$

and has the solution, $y(z) = J_m(z)$, which is regular at $z = 0$.

Midterm exam - solution

1. We separate variables: $u(r,\theta) = X(r)Y(\theta)$, giving $X(r) = r^m$ or r^{-m} and $Y(\theta) = \cos m\theta$ or $\sin m\theta$. Because $Y(\theta)$ is $2\pi-$periodic, $m = 0, 1, 2, ...$ We discard r^m as it is not regular for $r \to \infty$, and $\cos m\theta$ because the boundary condition demands that $u(r,\theta)$ is odd in θ. Hence

$$u(r,\theta) = \sum_{m=1}^{\infty} b_m r^{-m} \sin m\theta, \qquad b_m = \int_0^{\pi} \sin m\theta d\theta = [1 - (-1)^m]/m$$

We rewrite this solution as

$$u = -i \sum_{m>0, m \ odd} \frac{1}{m} [(r^{-1}e^{i\theta})^m - (r^{-1}e^{-i\theta})^m]$$

and use the summation for each term to obtain

$$u = -\frac{i}{2} \ln \left(\frac{r^2 - 1 + 2ir\sin\theta}{r^2 - 1 - 2ir\sin\theta} \right).$$

2. Dividing Bessel's equation with $m = 0$ and $y = J_0(z)$ by z and then integrating furnishes

$$zJ_0' + \int_0^z \hat{z} J_0(\hat{z}) d\hat{z} = 0,$$

which gives the first result in view of $J_1 = -J_0'$. Next we multiply the equation by J_0' and use

$$zJ_0'(zJ_0') = \frac{1}{2}[z^2(J_0')^2]' \qquad \text{and} \qquad z^2 J_0 J_0' = \frac{1}{2} z^2 [(J_0)^2]'$$

to arrive at the second result. Last, we integrate the second result in z

$$0 = z^2(J_1)^2 + \int_0^z \hat{z}^2 \frac{d}{d\hat{z}} [J_0(\hat{z})]^2 d\hat{z} = z^2(J_1)^2 + z^2(J_0)^2 - 2\int_0^z \hat{z}[J_0(\hat{z})]^2 d\hat{z},$$

giving the third result.

We now separate variables for the PDE, $u = X(r)T(t)$, finding

$$T_{tt} = -k^2 T, \qquad X_{rr} + \frac{1}{r} X_r + k^2 X = 0.$$

Hence $X = J_0(kr)$. Moreover, the boundary condition, $X(1) = 0$ implies that $J_0(k) = 0$, so k must be a zero of $J_0(z)$. Denote the n^{th} such zero by k_n. The solution so far is therefore

$$u(r,t) = \sum_{n=1}^{\infty} [a_n \cos(k_n t) + b_n \sin(k_n t)] J_0(k_n r)$$

We also have the initial conditions $u(r,0) = 0$ and $u_t(r,0) = 1$, and so $a_n = 0$ and

$$b_n = \frac{\int_0^1 J_0(k_n r) r dr}{k_n \int_0^1 [J_0(k_n r)]^2 r dr} = \frac{2}{k_n^2 J_1(k_n)},$$

using the results of the first part of the question.