Midterm exam - 2019

Closed book exam; no calculators. Adequately explain the steps you take.

1. Use separation of variables to solve

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

on the semicircle, $r \le 1$ and $0 \le \theta \le \pi$, subject to $u(r,0) = u(r,\pi) = 0$ and $u(1,\theta) = f(\theta)$. Recalling that $\sum_{n=1}^{\infty} z^n = z/(1-z)$, sum your series for $u(r,\theta)$, and hence write down a compact expression for the solution in terms of a single integral. Evaluate the integral if the boundary function is localized and $f(\theta) \approx \delta(\theta - \frac{1}{2}\pi)$, where $\delta(x)$ is Dirac's delta-function.

2. Given that $J_0 \approx 1 - \frac{1}{2}z^2 + ...$ and $J_1 \approx z + ...$ for $z \ll 1$, differentiate Bessel's equation for $m = 0$ to establish that $J_1(z) = -J_0'(z)$. Then show that

$$\int_0^z zJ_0(z)dz = zJ_1(z) \qquad \& \qquad \int_0^z z^2 J_1(z)dz = 2zJ_1(z) - z^2 J_0(z).$$

Use separation of variables to solve

$$u_{tt} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$$

inside the unit disk $r \le 1$ and $0 \le \theta \le 2\pi$, subject to $u(1,\theta,t) = u_t(r,\theta,0) = 0$ and $u(r,\theta,0) = r\sin\theta$. Express your result as a sum involving Bessel functions without any integrals.

Helpful information:

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Bessel's equation is

$$z^2y'' + zy' + (z^2 - m^2)y = 0,$$

and has the solution, $y(z) = J_m(z)$, which is regular at $z = 0$ and satisfies (for any real constant α)

$$\int_0^x x[J_m(\alpha x)]^2 dx = \frac{1}{2}x^2[J_m'(\alpha x)]^2 + \frac{1}{2}\left(x^2 - \frac{m^2}{\alpha^2}\right)[J_m(\alpha x)]^2.$$

Helpful trig identities:

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \qquad \& \qquad \sin(A \pm B) = \sin A\cos B \pm \cos A\sin B.$$

Midterm exam - solution

1. We separate variables: $u(r,\theta) = X(r)Y(\theta)$, giving $X(r) = r^m$ or r^{-m} and $Y(\theta) = \cos m\theta$ or $\sin m\theta$. Because $Y(0) = Y(\pi) = 0$ we must have the $\sin m\theta$ with $m = 1, 2, ...$ We discard r^{-m} as it is not regular for $r \to 0$. Hence

$$u(r,\theta) = \sum_{m=1}^{\infty} b_m r^m \sin m\theta, \qquad b_m = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin m\theta d\theta$$

We rewrite this solution as

$$u(r,\theta) = \frac{1}{\pi}\int_0^{\pi} f(\hat{\theta}) \sum_{m=1}^{\infty} r^m[\cos m(\theta - \hat{\theta}) - \cos m(\theta + \hat{\theta})]d\hat{\theta}$$

$$= \frac{1}{2\pi}\int_0^{\pi} f(\hat{\theta}) \sum_{m=1}^{\infty} [(re^{i(\theta-\hat{\theta})})^m + (re^{-i(\theta-\hat{\theta})})^m - (re^{i(\theta+\hat{\theta})})^m - (re^{-i(\theta+\hat{\theta})})^m]d\hat{\theta}$$

Hence, using the given sum,

$$u = \frac{1-r^2}{2\pi}\int_0^{\pi} f(\hat{\theta}) \left[\frac{1}{1+r^2-2r\cos(\theta-\hat{\theta})} - \frac{1}{1+r^2-2r\cos(\theta+\hat{\theta})}\right] d\hat{\theta}.$$

If $f = \delta(\theta - \pi/2)$, then

$$u = \frac{1-r^2}{2\pi}\left[\frac{1}{1+r^2-2r\sin\theta} - \frac{1}{1+r^2+2r\sin\theta}\right] = -\frac{r(1-r^2)\sin\theta}{\pi[(1+r^2)^2 - 4r^2\sin^2\theta]}.$$

2. Bessel's equation with $m = 0$ is $J_0'' + z^{-1}J_0' + J_0 = 0$. Differentiating gives $J_0''' + z^{-1}J_0'' + J_0' - z^{-2}J_0' = 0$, which is the ODE for $J_1(z)$. Therefore J_0' is proportional to $J_1(z)$. The expansion around $z = 0$ then establishes that $J_1 = -J_0'$.

Integrating $(zJ_0')' + zJ_0 = 0$ and using $J_0' = -J_1$ establishes the first integral relation. For the second,

$$\int z^2 J_1 dz = -\int z^2 J_0' dz = -z^2 J_0 + 2\int zJ_0 dz = -z^2 J_0 + 2zJ_1,$$

as required.

We now separate variables for the PDE, $u = X(r)Y(\theta)T(t)$, finding

$$T_{tt} = -k^2 T, \qquad Y_{\theta\theta} = -m^2 Y, \qquad X_{rr} + \frac{1}{r}X_r + \left(k^2 - \frac{m^2}{r^2}\right)X = 0.$$

Hence $X = J_1(kr)$, $Y = \sin\theta$ and $T = \cos(kt)$ given regularity at the origin and the initial conditions. Moreover, the boundary condition, $X(1) = 0$ implies that $J_1(k) = 0$, so k must be a zero of $J_1(z)$. Denote the n^{th} such zero by k_n. The solution is therefore

$$u(r,t) = \sum_{n=1}^{\infty} b_n \cos(k_n t) J_1(k_n r)\sin\theta,$$

with

$$b_n = \frac{\int_0^1 J_1(k_n r) r^2 dr}{\int_0^1 [J_1(k_n r)]^2 r dr} = -\frac{2J_0(k_n)}{k_n[J_1'(k_n)]^2},$$

using the results of the first part of the question and the helpful information.