Midterm exam - 2020

Closed book exam; no calculators. Adequately explain the steps you take and answer as much as you can (partial credit awarded).

1. Use separation of variables to solve

$$u_t = x^2 u_{xx} + x u_x + u_{yy}, \qquad 1 < x < e = \exp(1), \quad 0 < y < \pi,$$

subject to $u(x, 0, t) = u(x, \pi, t) = u(1, y, t) = u(e, y, t) = 0$ and $u(x, y, 0) = f(x, y)$. Be sure to state the specific form of the Sturm-Liouville problem for the $x-$dependence of the solution and correctly identify its weight function. Also, note that $x^{i\omega} \equiv e^{i\omega \ln x}$. Give an explicit solution if $f = 1$.

2. Use separation of variables to solve

$$u_t = u + \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$$

inside the unit disk $r \leq 1$ and $0 \leq \theta \leq 2\pi$, subject to $u(1, \theta, t) = 0$ and $u(r, \theta, 0) = \sin 2\theta$. Using the helpful information, express your result as a sum involving Bessel functions without any integrals.

Helpful information:

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

with

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

Bessel's equation is

$$z^2 y'' + z y' + (z^2 - m^2) y = 0,$$

and has the solution, $y(z) = J_m(z)$, which is regular at $z = 0$ and satisfies the relations

$$z[J_2(z) + J_0(z)] = 2J_1(z), \quad J_0'(z) = -J_1(z), \quad \frac{d}{dz}[z J_1(z)] = z J_0(z), \quad \int_0^z z J_0(z) dz = z J_1(z).$$

$$\int_0^z z [J_m(z)]^2 dz = \frac{1}{2} z^2 [J_m'(z)]^2 + \frac{1}{2} \left(z^2 - m^2\right) [J_m(z)]^2.$$

The **Sturm-Liouville ODE** is

$$[p(x) y']' + \lambda \sigma(x) y + q(x) y = 0, \qquad a < x < b,$$

with $\sigma(x) > 0$ and $p(x) > 0$. The associated expansion formula using the eigensolutions $\{\lambda_n, y_n(x)\}$ is

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad c_n = \frac{\int_a^b f(x) y_n(x) \sigma(x) dx}{\int_a^b [y_n(x)]^2 \sigma(x) dx}.$$

Helpful trig identities:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad \& \qquad \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

Midterm exam - solution

1. We separate variables: $u(r,\theta) = X(x)Y(y)T(t)$, giving

$$\frac{T'}{T} = \frac{x^2X'' + xX'}{X} + \frac{Y''}{Y},$$

giving $T \propto e^{-\lambda t}$ and $Y \propto \sin my$ with $m = 1, 2, ..$ (because $Y(0) = Y(\pi) = 0$). The ODE in x is an Euler equation with the form

$$x^2X'' + xX' + (\lambda - m^2)X = 0 \qquad \text{or} \qquad (xX')' + (\lambda - m^2)x^{-1}X = 0,$$

which is a Sturm-Liouville system with $p(x) = x$, $\sigma(x) = x^{-1}$ and eigenvalue $\omega^2 = \lambda - m^2 > 0$. The solutions are therefore $X \propto \sin(\omega \ln x)$ or $\cos(\omega \ln x)$. We discard the latter and conclude $\omega = n\pi$, $n = 1, 2, ...$, in view of the boundary conditions $X(1) = X(e) = 0$. Hence, the general solution is

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{mn} e^{-(m^2+n^2\pi^2)t} \sin my \sin(n\pi \ln x),$$

where

$$c_{mn} = \frac{2}{\pi}\int_0^{\pi}\int_1^{e} f(x,y) \sin my \sin(n\pi \ln x)\frac{1}{x}dxdy \left[\int_1^e \sin^2(n\pi \ln x)\frac{1}{x}dx\right]^{-1}$$

and the integral in the denominator equals $\frac{1}{2}$. For $f = 1$, we find

$$c_{mn} = \frac{4}{nm\pi^2}[1 - (-1)^m][1 - (-1)^n].$$

2. We separate variables for the PDE, $u = X(r)Y(\theta)T(t)$, finding

$$T_t = (1 - k^2)T, \qquad Y_{\theta\theta} = -m^2Y, \qquad r^2X_{rr} + rX_r + \left(k^2r^2 - m^2\right)X = 0.$$

But the initial condition is proportional to $\sin 2\theta$, so we may take $m = 2$ and $Y = \sin 2\theta$, giving $X(r) = J_2(kr)$ in view of regularity at the origin. The boundary condition $X(1) = 0$ then implies that $k = z_j$, one of the zeros of $J_2(z)$. The solution is therefore

$$u(r,t) = \sum_{n=1}^{\infty} c_n e^{(1-z_n^2)t} J_2(z_nr)\sin 2\theta,$$

with (given that the problem in r has Sturm-Liouville form with weight function $\sigma = r$)

$$c_n = \frac{\int_0^1 J_2(z_nr)rdr}{\int_0^1 [J_2(z_nr)]^2 rdr}.$$

But

$$\int_0^z zJ_2dz = 2\int_0^z J_1dz - \int_0^z zJ_0dz = 2J_0(0) - 2J_0(z) - zJ_1(z)$$

and

$$\int_0^z J_2^2 zdz = \frac{1}{2}z^2(J_2')^2 + \frac{1}{2}(z^2 - 4)J_2^2$$

from the helpful information, and so

$$c_n = \frac{2[2J_0(0) - 2J_0(z_n) - z_nJ_1(z_n)]}{z_n^2[J_2'(z_n)]^2}.$$